First Hour Exam (100 points)

1 (16 points). Find the general solution to the following system of three equations in four unknowns.

$$\begin{aligned} x_1 + 3x_2 - x_3 - 3x_4 &= 0 \\ 2x_1 + 6x_2 - x_3 - 4x_4 &= 1 \\ -x_1 - 3x_2 + 2x_3 + 4x_4 &= 2 \end{aligned}$$

Answer: $(x_1, x_2, x_3, x_4) = (0, 0, 3, -1) + t(-3, 1, 0, 0), \quad t \in \mathbb{R}$.

2 (7 points). Compute the projection of the vector $(5, 12)$ onto the vector $(3, 4)$.
Answer: $\text{Proj}_{(3,4)}(5, 12) = \frac{63}{25}(3, 4)$.

3 (28 points). Set up, but **DO NOT REDUCE**, (possibly augmented) matrices which could be used to deal with each of the following situations. Include justification to gain partial credit for incorrect answers.

a) finding all vectors which are orthogonal to both $(2, 1, 1)$ and $(-1, 1, 3)$, **Answer:** $\begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

b) determining if $(1, 2, 3)$ is a linear combination of $(2, 1, 1)$ and $(-1, 1, 3)$,
Answer: $\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{array}\right]$.

c) finding a cartesian equation for the plane given parametrically as
$\mathbf{x} = (1, 2, 3) + s(2, 1, 1) + t(-1, 1, 3)$, **Answer:** $\left[\begin{array}{cc|c} 2 & -1 & x_1 - 1 \\ 1 & 1 & x_2 - 2 \\ 1 & 3 & x_3 - 3 \end{array}\right]$.

d) finding a parabola which passes through the points $(1, 1), (2, 3)$, and $(5, 7)$.
Answer: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 3 \\ 25 & 5 & 1 & 7 \end{array}\right]$.

4 (14 points). Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in $\mathbb{R}^5$.

a) Define what it means for a vector $\mathbf{b} \in \mathbb{R}^5$ to be a *linear combination* of $\mathbf{v}_1$ and $\mathbf{v}_2$. **Answer:** $\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ for some scalars $c_1, c_2 \in \mathbb{R}$.

b) Prove that if a vector $\mathbf{a} \in \mathbb{R}^5$ is simultaneously orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$, then $\mathbf{a}$ is also orthogonal to every linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
Answer: $\mathbf{a} \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1(\mathbf{a} \cdot \mathbf{v}_1) + c_2(\mathbf{a} \cdot \mathbf{v}_2) = 0$.

5 (35 points). Provide examples of the following. No justification is required.

a) a unit vector in $\mathbb{R}^2$ which is orthogonal to $(2,-3)$, **Answer:** $\frac{1}{\sqrt{13}}(3,2)$.

b) an echelon matrix which is not in reduced echelon form, **Answer:** The one by one matrix $[2]$.

c) a direction vector $\mathbf{v}$ such that the parametric equations $\mathbf{x} = (1,2,0) + t\mathbf{v}$ and $\mathbf{x} = (0,0,1) + t\mathbf{v}$ represent the same line, **Answer:** $\mathbf{v} = (1,2,-1)$ or any non-zero scalar multiple of that vector.

d) a matrix A such that the equation $A\mathbf{x} = \mathbf{b}$ does not have a unique solution for any vector $\mathbf{b}$, **Answer:** Any matrix with more columns than rows.

e) a two-by-two matrix A such that $A^3 = I$ but $A \neq I$. **Answer:** $\begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}$.