CHAPTER 4

2) (SNEAKY PROOF DUE TO MR. COWART)
Suppose $f: X \to Y$ is continuous and $E \subseteq X$.
Then $f(E) \subseteq \overline{f(E)}$ so $E \subseteq f^{-1}(f(E)) \subseteq f^{-1}(\overline{f(E)})$.
Since the rightmost set is closed, we actually have $\overline{E} \subseteq f^{-1}(\overline{f(E)})$ i.e. $f(\overline{E}) \subseteq \overline{f(E)}$ as desired.

3) $Z_f = f^{-1}(\{0\})$ is the inverse image of a closed set.
[REFERENCE TO $f(Z_f)$ is unnecessary and distracting.]

4) a) If E is dense in X, then $f(X) = f(\overline{E}) \subseteq \overline{f(E)}$ by Problem 2 so $f(E)$ is indeed dense in $f(X)$.

b) You can either use sequences or letting $x \in X$ and $\varepsilon > 0$ be given, find $p \in E$ with $d(f(x), f(p)) < \varepsilon/2$ and $d(g(x), g(p)) < \varepsilon/2$. Since $f(p) = g(p)$, the Δ-inequality yields $d(f(x), g(x)) < \varepsilon$. The arbitrariness of ε means $d(f(x), g(x)) = 0$ whence $f(x) = g(x)$.

6) $f: E^{\text{compact}} \to Y$. (a) If f is continuous, so is $F: E \to E \times Y$ by $F(x) = (x, f(x))$. But then graph $f = F(E)$ is the continuous image of a compact set. [If you try a direct pf instead, you must keep in mind the fact that not all open sets in $E \times Y$ are rectangles].

(b) If the graph of f is compact, fix $a \in E$ and $\varepsilon > 0$. Take $V_0 = \{(x,y) \in E \times Y : d(y, f(a)) < \varepsilon\}$ and for each $n = 1, 2, \ldots$, set $V_n = \{(x,y) \in E \times Y : d(x,a) > \frac{1}{n}\}$. If $V_0 \ldots V_N$ cover gr(f), then $d(x,a) < \frac{1}{N} \Rightarrow d(f(x), f(a)) < \varepsilon$.