PRINT NAME: ______________________________
Student ID#: ______________________________

Mathematics 2200 - Quiz Six
Wednesday, October 29, 2003

1. Find the derivatives of the following functions: (a). $f(x) = e^{e^x}$. (b) $g(x) = \left(1 + \frac{1}{x}\right)^x$.

Solution: (a). Just apply the chain rule and obtain $\frac{dy}{dx} = e^{e^x} e^x = e^{x+e^x}$.
(b). You need to use the method of Logarithmic differentiation. Take the natural logarithms of both side, we have

$$\ln g(x) = \ln\left[\left(1 + \frac{1}{x}\right)^x\right] = x \ln\left(1 + \frac{1}{x}\right) = x \ln\left(\frac{1+x}{x}\right) = x[\ln(1+x) - \ln x].$$

Differentiate with respect to x,

$$\frac{1}{g(x)} g'(x) = [\ln(1+x) - \ln x] - x\left(\frac{1}{1+x} - \frac{1}{x}\right) = [\ln(1+x) - \ln x] + \frac{1}{1+x}.$$

Multiply both sides by $g(x)$:

$$g'(x) = \left(1 + \frac{1}{x}\right)^x \left[\ln(1+x) - \ln x + \frac{1}{1+x}\right].$$

□

2. Find the derivative of the function $y = e^{-x} \sin x$ and the critical points of the function for $x > 0$.

Solution: Find the derivative by applying the product rule:

$$\frac{dy}{dx} = -e^{-x} \sin x + e^{-x} \cos x = e^{-x}(-\sin x + \cos x).$$

Set the derivative equal to zero and obtain

$$-\sin x + \cos x = 0, \quad \cos x = \sin x, \quad \tan x = 1; \quad x = k\pi + \frac{\pi}{4} \quad \text{for } k = 0, 1, 2, \cdots.$$

□

3. Find the equation of the line tangent to the graph of the given function at the indicated point: (a). $y = \frac{\ln x}{x^3}$ at the point (e, e^{-3}). (b). $y = x^2 \ln x$ at the point $(1, 0)$.

Solution: (a). Write the function as $y = x^{-3} \ln x$. Then apply the product rule to find the derivative

$$\frac{dy}{dx} = -3x^{-4} \ln x + x^{-3} \cdot \frac{1}{x} = -3x^{-4} \ln x + x^{-4} = x^{-4}(1 - 3lnx).$$

It is easy to see that the point (e, e^{-3}) is on the curve. Let $x = e$ in the derivative and find

$$\left.\frac{dy}{dx}\right|_{x=e} = -2e^{-4}.$$

And the equation of the tangent line is

$$y - e^{-3} = -2e^{-4}(x - e), \quad \text{hence} \quad y + 2e^{-4}x = 3e^{-3}.$$

(b) Apply the product rule and find

$$\frac{dy}{dx} = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x = x(1 + 2 \ln x).$$

The point $(1, 0)$ is on the curve. Set $x = 1$ in the derivative

$$\left.\frac{dy}{dx}\right|_{x=1} = 1.$$

The equation of the tangent line is

$$y - 0 = 1 \cdot (x - 1), \quad y = x - 1.$$

□