Mathematics 2200 - Test Three
Friday, Nov. 7, 2003

1. Find the derivatives of the following functions.

(a) $y = \ln\left(\frac{x^2-1}{x^2+1}\right)$

Solution: Note that $y = \ln\left(\frac{x^2-1}{x^2+1}\right) = \ln(x^2-1) - \ln(x^2+1)$. Hence

$$\frac{dy}{dx} = \frac{1}{x^2-1}(x^2-1)' - \frac{1}{x^2+1}(x^2+1)' = \frac{2x}{x^2-1} - \frac{2x}{x^2+1} = \frac{4x}{x^4-1}.$$

□

(b) $g(x) = (2x^2 - 3x)e^{-x}$

Solution: You just need to use the product rule and note that $(e^{-x})' = -e^{-x}$.

$$\frac{dg}{dx} = (4x-3)e^{-x} - (2x^2-3x)e^{-x} = e^{-x}(-2x^2+7x-3).$$

□

(c) $f(x) = (\ln x)^{\ln x}$

Solution: This problem is very close to the problem in the last quiz. Note that

$$\ln f(x) = \ln(\ln x)^{\ln x} = \ln x \cdot \ln(\ln x).$$

Hence

$$\frac{1}{f(x)}\frac{df}{dx} = \frac{1}{x}\ln(\ln x) + \ln x \frac{1}{\ln x}\frac{1}{x} = \frac{1+\ln(\ln x)}{x}.$$

Hence $\frac{df}{dx} = (\ln x)^{\ln x}\frac{1+\ln(\ln x)}{x}$. □

(d) $y = (2+e^x)^{e^x}$.

Solution: This problem is also from the last quiz.

$$\ln y = \ln[(2+e^x)^{e^x}] = e^x \ln(2+e^x).$$

Hence

$$\frac{1}{y}\frac{dy}{dx} = e^x\ln(2+e^x) + e^x\frac{e^x}{2+e^x}.$$

We used the product rule and $(e^x)' = e^x$ and $(\ln x)' = 1/x$. Therefore

$$\frac{dy}{dx} = (2+e^x)^{e^x} e^x\left[\ln(2+e^x) + \frac{e^x}{2+e^x}\right].$$

□

2. Suppose that water is emptied from a spherical tank of radius 10ft. If the depth of the water in the tank is 4ft and is decreasing at rate of 2ft/s, at what rate is the radius r of the top surface of the water decreasing?

Solution: This is the only problem we did not do in the class and but this is a homework problem. Refer to the picture in the book, we know $\frac{dy}{dt} = -2, \quad$ when $\quad y = 4$ and want to find $\frac{dr}{dt}$ when $y = 4$. So we need to find the relationship of y and r. But this is obvious: $(10-y)^2 + r^2 = 10^2$. Hence $r^2 = 20y - y^2$ and $r = (20y - y^2)^{1/2}$. Differentiate both sides with respect to t, we have

$$\frac{dr}{dt} = \frac{10-y}{(20y-y^2)^{1/2}} \frac{dy}{dt} = \frac{10-4}{(80-16)^{1/2}}(-2) = -\frac{3}{2}.$$

□

3. A water trough is to be made from a long strip of tin 6 ft wide by bending up at an angle θ a 2-ft strip on each side. What angle θ would maximize the cross-sectional area?

Solution: The area of the cross-section in terms of θ is

$$A(\theta) = \frac{1}{2} \cdot 2\sin\theta(2 + 2 + 2 \cdot 2\cos\theta) = 4\sin\theta + 4\sin\theta\cos\theta.$$

The domain of the function is $0 \le \theta \le 2\pi/3$ with
Differentiate with respect to θ and we find

$$A(\theta) = 4(\cos\theta + \cos^2\theta - \sin^2\theta) = 4(2\cos^2\theta + \cos\theta - 1)$$

here we have used $\sin^2\theta = 1 - \cos^2\theta$. $A'(\theta) = 0$ implies that $\cos\theta = \frac{1}{2}$ and $\cos\theta = -1$. This yields $\theta = \frac{\pi}{3}$ and $\theta = \pi$. But $0 \le \theta \le \frac{2\pi}{3}$,
$\theta = \pi$ is NOT in the domain. Note that $A(0) = 0$, $A(\frac{2\pi}{3}) = \sqrt{3}$ and

$$A(\frac{\pi}{3}) = 3\sqrt{3}.$$

Hence we find the maximal value of the cross-section area. □

4. Find the equation of the line tangent to the curve $x \sin y + y \sin x = \pi$ at the point $(\frac{\pi}{2}, \frac{\pi}{2})$.

Solution: This problem tests if you have mastered the material about the implicit differentiation. YOU SHOULD ALWAYS REMEMBER THAT y is a FUNCTION of x. Differentiate with respect to x, we have

$$\sin y + x \cos y \frac{dy}{dx} + \frac{dy}{dx} \sin x + y \cos x = 0.$$

π is a constant so its derivative is zero. Note that the point $(\frac{\pi}{2}, \frac{\pi}{2})$ is on the curve. In order to find the equation of the tangent line, we need to know the slope y' when $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}$. So we let $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}$ in the equation

$$\sin y + x \cos y \frac{dy}{dx} + \frac{dy}{dx} \sin x + y \cos x = 0.$$

and obtain $1 + \frac{dy}{dx} = 0$ because $\cos \frac{\pi}{2} = 0$. Hence $y' = -1$ at point $(\frac{\pi}{2}, \frac{\pi}{2})$. The equation of the tangent line is

$$y - \frac{\pi}{2} = -(x - \frac{\pi}{2}), \quad \text{hence } x + y = \pi$$

□

5. * **Bonus question** (Note: This is not an easy question and will marked very strictly) Find all the values of the constant c so that the two curves $y = cx^2$ and $y = \ln x$ have exactly one point in common.

Solution: First when $c = 0$, $y = cx^2 = 0$ and $y = \ln x$ have exactly one point in common $(1, 0)$ because $y = 0 = \ln x$ implies $x = 1$.

Second we note that when $c < 0$, the curve $y = cx^2$ and $y = \ln x$ always have exactly one point in common. This can be shown by the follow argument. Set $f(x) = \ln x - cx^2$, then $f(1) = -c > 0$ when $c < 0$ and $f(e^c) = c - ce^{2c} = c(1 - e^{2c}) < 0$. The second is true because $e^{2c} < 1$ when $c < 0$. Then by the Intermediate Value Property of the continuous function there exist x_0, $e^c < x_0 < 1$ such that $f(x_0) = 0$. And this implies $cx_0^2 = \ln x_0$. Hence the two curves $y = cx^2$ and $y = \ln x$ have at least one point in common, that is $(x_0, \ln x_0)$. Next we need to show this is only point in common. Note that $f'(x) = \frac{1}{x} - 2cx > 0$ for all $x > 0$. Hence $f(x)$ is an one-to-one function. This proves that x_0 is the only number such that $f(x_0) = 0$.

Third we consider the case $c > 0$. That the two curves $y = cx^2$ and $y = \ln x$ have exactly one point in common implies they are tangent to each other at the intersection point. Hence

$$cx^2 = \ln x \quad \text{and} \quad 2cx = \frac{1}{x}$$

at the point of the intersection. The second equation implies $2cx^2 = 1$, hence $cx^2 = \frac{1}{2}$, Substitute this into $cx^2 = \ln x$ and we have $\ln x = \frac{1}{2}$ and $x = e^{1/2}$. Therefore $c = \frac{1}{2x^2} = \frac{1}{2e}$.

□