Mathematics 2210 - Quiz Four
Wednesday, October 7, 2004

1. Sketch the region bounded by the given curves, then find the volume of the bodies which were generated by rotating around y-axis.

   a) $x = y^2, \quad x = 2 - y.$

   SOLUTION: The region is defined by two functions with y as independent variable. Their intersection points are given by $y^2 = 2-y$, and this yields $y^2+y-2=0$ and $y = -2$ and $y = 1$. Hence we use the method of cross section to find the volume:

$$\begin{aligned} V &= \int_{-2}^{1} \pi[(2-y)^2 - y^4]dy \\ &= \pi(\frac{(y-2)^3}{3} - \frac{y^5}{5})\Big|_{-2}^{1} \\ &= 14.4\pi. \end{aligned}$$

□

   b) $y = x^2, \quad y = 4(x-1)^2.$

   SOLUTION: The intersection points are $x = 2/3$ and $x = 2$. We use the method of cylinder shell to find the volume:

$$\begin{aligned} V &= \int_{2/3}^{2} 2\pi x[x^2 - 4(x-1)^2]dx \\ &= 2\pi \int_{2/3}^{2} (-3x^3 + 8x^2 - 4x)dx \\ &= 2\pi(-\frac{3}{4}x^4 + \frac{8}{3}x^3 - 2x^2)\Big|_{2/3}^{2} \\ &= \frac{256\pi}{81}. \end{aligned}$$

□

2. Find the total length of the astroid whose graph is given by equation $x^{2/3} + y^{2/3} = 1$.

   SOLUTION: Note that the curve is symmetric and we just need to find the length for x from 0 and 1.

$$y = (1 - x^{2/3})^{3/2}, \quad \frac{dy}{dx} = \frac{3}{2}(1 - x^{2/3})^{1/2}(-\frac{2}{3}x^{-1/3}) = -x^{-1/3}(1 - x^{2/3})^{1/2}.$$

   Hence

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^{-2/3}(1 - x^{2/3}) = x^{-2/3}.$$

   and

$$L = 4\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx = 4\int_0^1 x^{-1/3}dx = 6x^{2/3}\Big|_0^1 = 6.$$

□