Mathematics 2210 - Test One
Friday, September 17, 2004

1. a) (20) Use the Riemann sums to evaluate the integral $\int_0^4 (x^3 - 4x)dx$. You may need the formulae $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ and $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

   b) (10) Use the Fundamental Theorem of Calculus to evaluate the integral to verify your result.

   c) (20) Find the area of regions bounded by the function $y = x^3 - 4x$ and the line $y = 5x$.

SOLUTION: This problem is taken from the example of the text book. It is very close to example 3 in section 5.4. I would like to remind you to read the textbook, specially, the examples.

a). With $a = 0$ and $b = 4$, we have $\Delta x = 4/n$ and $x_i = 4i/n$. Hence

$$\begin{aligned}
\int_0^4 (x^3 - 4x)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{4i}{n}\right)^3 - 4\left(\frac{4i}{n}\right) \right] \cdot \frac{4}{n} \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left[\frac{64i^3}{n^3} - \frac{16i}{n} \right] \\
&= \lim_{n\to\infty} \left[\frac{256}{n^4} \sum_{i=1}^{n} i^3 - \frac{64}{n^2} \sum_{i=1}^{n} i \right].
\end{aligned}$$

We now use the formulae about the sum $\sum i^3$ and $\sum i$ to convert each of the last two sums to closed form:

$$\begin{aligned}
\int_0^4 (x^3 - 4x)dx &= \lim_{n\to\infty} \left[\frac{256}{n^4} \cdot \frac{n^2(n+1)^2}{4} - \frac{64}{n^2} \cdot \frac{n(n+1)}{2} \right] \\
&= \lim_{n\to\infty} \left[\frac{64}{n^2}(n+1)^2 - \frac{32}{n}(n+1) \right] \\
&= \lim_{n\to\infty} \left[64\left(1 + \frac{1}{n}\right)^2 - 32\left(1 + \frac{1}{n}\right) \right] \\
&= 32.
\end{aligned}$$

b). We can use the Fundamental Theorem of Calculus to find

$$\int_0^4 (x^3 - 4x)dx = \frac{1}{x^4} - 2x^2 \Big|_0^4 = 64 - 32 = 32.$$

c). The intersection points of the curve $y = x^3 - 4x$ and $y = 5x$ are determined by the equation:

$$y = x^3 - 4x \quad \text{and} \quad y = 5x, \quad \text{hence} x^3 - 4x = 5x,\ x^3 - 9x = 0.$$

This yields $x = 0$, $x = 3$ and $x = -3$. For x from -3 to 0, the curve is above the line and for x from 0 to 3, the curve is below the line. Hence

$$A = \int_{-3}^{0} (x^3 - 4x - 5x)dx + \int_0^3 (5x - x^3 + 4x)dx = 2\int_0^3 (9x - x^3)dx = \frac{81}{2}.$$

□

2. Using the method of substitution to find the integrals

(a) (10) $\int \frac{x^5}{\sqrt{x^3+9}} dx.$

SOLUTION: Let the substitution be $u = x^3 + 9$, then $du = 3x^2$ and $x^3 = u - 9$.

$$\int \frac{x^5}{\sqrt{x^3+9}} dx = \frac{1}{3} \int \frac{u-9}{\sqrt{u}} du = \frac{1}{3} \int (u^{1/2} - 9u^{-1/2}) = \frac{2}{9} u^{3/2} - 6u^{1/2} + C.$$

Put back the variable $u = x^3 + 9$, we have

$$\int \frac{x^5}{\sqrt{x^3+9}} dx = \frac{2}{9}(x^3+9)^{3/2} - 6(x^3+9)^{1/2} + C.$$

□

(b) (10) $\int_0^{\frac{\pi}{2}} 2\sin^3 x \cos x dx.$

SOLUTION: This is also an example from the textbook. Let $u = sinx$, then $du = \cos x dx$ and

$$\int_0^{\frac{\pi}{2}} 2\sin^3 x \cos x dx = \in 0^1 2u^3 du = \frac{1}{2} u^4 \bigg|_0^1 = \frac{1}{2}.$$

□

(c) (10) $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^3}.$

SOLUTION: Let $u = \sqrt{x} + 1$, then $du = \frac{1}{2\sqrt{x}} dx$,

$$\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^3} = 2\int \frac{du}{u^3} = -\frac{1}{u^2} + C = -(1+\sqrt{x})^{-2} + C.$$

□

3. a) (10) Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

SOLUTION: We first find the intersection points of the parabolas by solving their equations simultaneously. This gives $x^2 = 2x - x^2$, or $2x^2 = 2x$. Thus $x(x-1) = 0$, so $x = 0$ or 1. The points of intersection are $(0,0)$ and $(1,1)$. The top and bottom boundaries are $y_T = 2x - x^2$ and $y_B = x^2$. The area of the region is

$$A = \int_0^1 (2x - x^2 - x^2) dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

□

b) (10) Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$.

SOLUTION: The points of intersection occur when $\sin x = \cos x$, that is when $x = \frac{\pi}{4}$ (since $0 \le x \le \pi/2$). Observe that $\cos x \ge \sin x$ when $0 \le x \le \pi/4$ but $\sin x \ge \cos x$ when $\pi/4 \le x \le \pi/2$. Therefore, the required area is

$$A = \int_0^{\pi/2} |\cos x - \sin x| dx = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx.$$

Compute the two integrals, we find the area is $2\sqrt{2} - 2$. □

4. * **Bonus Problem (very difficult)** Evaluate the limit

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{n+n}} \right).$$

SOLUTION: We can write the limit as a definite integral. Note that

$$\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{n+n}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{1+\frac{i}{n}}}.$$

Take the interval to be $[0, 1]$ and the function to be $f(x) = 1/\sqrt{1+x}$. Then the limit is the definite integral

$$\int_0^1 \frac{1}{\sqrt{1+x}} dx = 2\sqrt{1+x}\Big|_0^1 = 2\sqrt{2} - 2.$$

□