Mathematics 2200 - Quiz Three
Monday, Sept. 26, 2005

1. Consider the function $f(x) = \dfrac{x}{1-2x}$.

   (1) Apply the definition of the derivative to find $f'(x)$.

   (2) Apply the quotient rule to verify your solution.

   (3) Find the equation of the line tangent to $y = f(x)$ with slope equal to 1.

   Solution: This is the problem 19 in section 3.1. I hope you have done your home work.

$$\begin{aligned} f(x+h) - f(x) &= \frac{x+h}{1-2(x+h)} - \frac{x}{1-2x} \\ &= \frac{(x+h)(1-2x) - x(1-2x-2h)}{(1-2x-2h)(1-2x)} \\ &= \frac{x(1-2x) + h - 2xh - x(1-2x) + 2xh}{(1-2x-2h)(1-2x)} \\ &= \frac{h}{(1-2x-2h)(1-2x)} \end{aligned}$$

   Hence

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{(1-2x-2h)(1-2x)} = \frac{1}{(1-2x)^2}$$

   Next we apply the quotient rule to verify that

$$f'(x) = \frac{1\cdot(1-2x) - x\cdot(-2)}{(1-2x)^2} = \frac{1-2x+2x}{(1-2x)^2} = \frac{1}{(1-2x)^2}.$$

   Set $f'(x) = 1$, we get $(1-2x)^2 = 1$ and This implies $1 - 2x = \pm 1$ and $x = 0$ or $x = 1$. So there are two points on the curve that the slope of the tangent line equals to 1 at these points. These points are $(0, 0)$ and $(1, -1)$. So the equations of the line tangent to the curve at $(0, 0)$ is $y = x$, and the equation of the line tangent to the curve at $(1, -1)$ is $y + 1 = x - 1$. □

2. Using the quotient rule to find the derivative of the function $f(x) = \dfrac{x^3}{x^2+1}$ and also find when the derivative equals to zero.

   Solution: We can use the quotient rule to find the derivative

$$f'(x) = \frac{3x^2(x^2+1) - x^3\cdot 2x}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2}$$

   $f'(x) = 0$ implies $x^2(x^2+3) = 0$. This implies $x = 0$. □

3. Find the equation of the tangent line that passes through $(1, 5)$ and is tangent to the curve $y = x^3$.

Solution: Since the point $(1, 5)$ is not on the curve. Suppose the tangent line is tangent to the curve at point (a, a^3). Then the equation of the tangent line is $y - a^3 = 3a^2(x - a)$. Insert $x = 1$ and $y = 5$ into the line equation we have $5 - a^3 = 3a^2(1 - a)$. Hence $2a^3 - 3a^2 + 5 = 0$. Then $a = -1$. The tangent line is $y = 3x + 2$. $\square$