Mathematics 2200 - Test One - Solution
Friday, September 16, 2005

1. Evaluate the following limits

(a) $\lim_{x\to 1} \frac{x^2+2x-3}{x^3-1}$.

Solution: We can just factor out the numerator and denominator and then cancel the common term $(x-1)$.

$$\lim_{x\to 1} \frac{x^2+2x-3}{x^3-1} = \lim_{x\to 1} \frac{(x-1)(x+3)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{x+3}{x^2+x+1} = \frac{4}{3}.$$

□

(b) $\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4}$

Solution: This problem needs to multiply the term $\sqrt{x+5}+3$.

$$\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} \cdot \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} = \lim_{x\to 4} \frac{(x+5)-9}{(x-4)(\sqrt{x+5}+3)} = \lim_{x\to 4} \frac{1}{\sqrt{x+5}+3} = \frac{1}{6}.$$

□

(c) $\lim_{x\to 0} \frac{1-\cos 3x}{x\sin x}$

Solution: We have done a similiar problem in the class.

$$\lim_{x\to 0} \frac{1-\cos 3x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos 3x}{x\sin x} \cdot \frac{1+\cos 3x}{1+\cos 3x} = \lim_{x\to 0} \frac{1-\cos^2(3x)}{x\sin x(1+\cos 3x)} = \lim_{x\to 0} \frac{\sin^2(3x)}{x\sin x(1+\cos 3x)} = \lim_{x\to 0} \frac{\sin 3x}{x} \cdot \frac{\sin 3x}{\sin x} \cdot \frac{1}{1+\cos 3x} = \frac{9}{2}.$$

□

(d) $\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{\sqrt{4+x}} - \frac{1}{2}\right)$

Solution: this is almost the same problem in the second quiz. If you don't get this problem, you need HELP!!!

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{\sqrt{4+x}}-\frac{1}{2}\right) &= \lim_{x\to 0}\frac{1}{x}\left(\frac{2-\sqrt{4+x}}{2\sqrt{4+x}}\right) \\
&= \lim_{x\to 0}\frac{2-\sqrt{4+x}}{2x\sqrt{4+x}}\cdot\frac{2+\sqrt{4+x}}{2+\sqrt{4+x}} \\
&= \lim_{x\to 0}\frac{4-(4+x)}{2x\sqrt{4+x}(2+\sqrt{4+x})} \\
&= \lim_{x\to 0}\frac{-x}{2x\sqrt{4+x}(2+\sqrt{4+x})} = \lim_{x\to 0}\frac{-1}{2\sqrt{4+x}(2+\sqrt{4+x})} \\
&= -\frac{1}{16} \quad \text{quotient, product and substitution laws.}
\end{aligned}$$

□

(e) $\lim_{x\to 2}\frac{\sqrt{2x}-\sqrt{x+2}}{x^2-2x}$

Solution: this problem is supposed to be difficult but it turns out to be very simple.

$$\begin{aligned}
\lim_{x\to 2}\frac{\sqrt{2x}-\sqrt{x+2}}{x^2-2x} &= \lim_{x\to 2}\frac{\sqrt{2x}-\sqrt{x+2}}{x(x-2)}\cdot\frac{\sqrt{2x}+\sqrt{x+2}}{\sqrt{2x}+\sqrt{x+2}} \\
&= \lim_{x\to 2}\frac{2x-(x+2)}{x(x-2)(\sqrt{2x}+\sqrt{x+2})} \\
&= \lim_{x\to 2}\frac{1}{x(\sqrt{2x}+\sqrt{x+2})} = \frac{1}{8}
\end{aligned}$$

□

2. A rectangular storage container with an open top has volume $\frac{1}{6}$ m^3. The length of its base is twice its width. Let x to be its width.

   (a) Show that its total surface area S (four sides plus the base) can be expressed as a function of the width x as follows: $S = f(x) = 2x^2 + \frac{1}{2x}$.

   (b) Using the definition of the slope-predictor $m(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ to find $m(a)$.

   (c) Find the value of a such that $m(a) = 0$.

Solution: (a). This problem is difficult. Let y be the height of the storage container. The length of its base is $2x$. Hence $2x\cdot x\cdot y = 1/6$. Its total surface area is

$$S = 2x\cdot x + 2\cdot 2x\cdot y + 2\cdot x\cdot y = 2x^2 + 6xy$$

where the first term is the area of the base, the second term is the area of the front and back, and the third term is the area of both sides. Then we solve for y from

$2x^2y = 1/6$ to get $y = \frac{1}{12x^2}$. Next substitute y into the formula for S and obtain $S = f(x) = 2x^2 + \dfrac{1}{2x}$.

(b) We first compute the term $f(a+h) - f(a)$:

$$\begin{aligned} f(a+h) - f(a) &= 2(a+h)^2 + \frac{1}{2(a+h)} - 2a^2 - \frac{1}{2a} \\ &= 2[(a+h)^2 - a^2] + \frac{1}{2a+2h} - \frac{1}{2a} \\ &= 2(2ah + h^2) + \frac{2a - (2a+2h)}{2a(2a+2h)} \\ &= 4ah + 2h^2 - \frac{h}{2a(a+h)} = h\left[4a + 2h - \frac{1}{2a(a+h)}\right] \end{aligned}$$

Hence we have

$$m(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0}\left[4a + h - \frac{1}{2a(a+h)}\right] = 4a - \frac{1}{2a^2}.$$

(c) Set $m(a) = 0$, this implies

$$4a - \frac{1}{2a^2} = 0, \quad 8a^3 = 1, \quad a^3 = \frac{1}{8}, \quad a = \frac{1}{2}$$

□

3. Find the value of the constant c so that the function $f(x) = \begin{cases} x^2 - c^2 & \text{if } x \le 6 \\ cx + 20 & \text{if } x > 6 \end{cases}$ is continuous for all x. Explain your answer.

Solution: If $x > 6$, then $f(x) = cx + 20$ and this is a linear function. Hence $f(x)$ is continuous.

If $x < 6$, then $f(x) = x^2 - c^2$ and this is a polynomial of degree 2. So it is continuous. We need to find the value of the constant c such that $f(x)$ is continuous at $x = 6$. Note that $f(6) = 36 - c^2$. $f(x)$ is continuous at $x = 6$ if and only if $\lim_{x\to 6^-} f(x) = \lim_{x\to 6^+} f(x) = f(6)$. This implies $36 - c^2 = 6c + 20$. This implies $c^2 + 6c - 16 = 0$. Hence $c = -8$ or $c = 2$.

□

4. * **Bonus Problem** (**very difficult**) Evaluate $\lim_{x\to 0} \dfrac{\sqrt[3]{1+cx} - 1}{x}$, where c is a constant.

Solution: This is a truly difficult problem. The trick to find this limit is to introduce the new variable $t = \sqrt[3]{1+cx}$. Then $t^3 = 1 + cx$ and $x = \frac{t^3-1}{c}$. Also we note that $\lim_{x\to 0} t = \lim_{x\to 0} \sqrt[3]{1+cx} = 1$. Hence

$$\lim_{x\to 0} \frac{\sqrt[3]{1+cx} - 1}{x} = \lim_{t\to 1} \frac{t-1}{\frac{t^3-1}{c}} = \lim_{t\to 1} \frac{c}{t^2 + t + 1} = \frac{c}{3}.$$

□