Mathematics 2200 - Test two
Wednesday, October 12, 2005

1. Use any suitable method to find $\frac{dy}{dx}$ for the following. There is no need to simplify your final answers for (a)-(d).

(a) $y = \frac{1}{\left(x^2 + \frac{1}{x^2}\right)^3}$

Solution: we can write the function as $y = \left(x^2 + x^{-2}\right)^{-3}$. Then we apply the chain rule and obtain

$$\frac{dy}{dx} = -3(x^2 + x^{-2})^{-4}(2x - 2x^{-3}).$$

$\square$

(b) $y = \frac{u^2}{u^2+1}$ and $u = \frac{x^2}{x^2+1}$

Solution: This is the problem from the quiz. First apply the quotient rule and obtain:

$$\frac{dy}{du} = \frac{2u(1+u^2) - u^2 \cdot 2u}{(1+u^2)^2} = \frac{2u}{(1+u^2)^2}$$

Also note that y as a function of u and u as a function of x are the same, so

$$\frac{du}{dx} = \frac{2x}{(1+x^2)^2}.$$

This implies:

$$\frac{dy}{dx} = \frac{2u}{(1+u^2)^2}\frac{2x}{(1+x^2)^2} = \frac{4x^3(1+x^2)}{(1+2x^2+2x^4)^2}.$$

$\square$

(c) $y = \sqrt{x^3 + \sqrt{x^2 + \sqrt{x}}}$

Solution: This problem seems difficult but it is really easy if you just follow the chain rule from outside to inside. We write the function as

$$y = \left\{x^3 + \left[x^2 + x^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}.$$

Then its derivative is

$$\frac{dy}{dx} = \frac{1}{2}\left\{x^3 + \left[x^2 + x^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}^{-\frac{1}{2}} \cdot \left\{3x^2 + \frac{1}{2}\left[x^2 + x^{\frac{1}{2}}\right]^{-\frac{1}{2}} \cdot \left[2x + \frac{1}{2}x^{-\frac{1}{2}}\right]\right\}$$

$\square$

(d) $y = \left[x - \left(1 - \frac{1}{x}\right)^{-1}\right]^2$.

Solution: You should first simplify the function inside the [ ].

$$f(x) = \left(x - 1 - \frac{1}{x-1}\right)^2.$$

Then we apply the chain rule and obtain

$$f'(x) = 2\left(x - 1 - \frac{1}{x-1}\right)\left(1 + \frac{1}{(x-1)^2}\right).$$

□

(e) Find the all points on the curve $y = \frac{1}{2\sin^2 x + 3\cos^2 x}$ where the tangent line is horizontal.

Solution: This problem is directly taken from the home work.

$$\frac{dy}{dx} = -\frac{4\sin x\cos x - 6\cos x\sin x}{(3\sin^2 x + 2\cos^2 x)^2} = \frac{2\sin x\cos x}{(3\sin^2 x + 2\cos^2 x)^2}.$$

To find the points on the curve where the tangent line is horizontal, we set $\frac{dy}{dx} = 0$. This implies that $2\sin x\cos x = 0$ and $\sin x = 0$ or $\cos x = 0$. Hence, $x = k\pi$ or $k\pi + \pi/2$ for k integers. When $x = k\pi$, $y = 1/2$ and when $x = k\pi + \pi/2$, $y = 1/3$. Therefore the tangent lines are horizontal at points $(k\pi, \frac{1}{3})$ and $(k\pi + \frac{\pi}{2}, \frac{1}{2})$. □

2. Find the maximum and minimum values attained by the given functions on the indicated closed intervals:
(a) $g(x) = x^4 - 4x^3 + 4x^2 + 2$; $[0, 4]$.

Solution: First find the derivative:

$$g'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2).$$

The critical points are $x = 0, 1, 2$. We need to find the values of the function at these points and the boundary points.

$$f(0) = 2,\ f(1) = 3,\ f(2) = 2 \text{ and } g(4) = 66$$

Hence the minimum is 2 and the the maximum is 66. □

(b) $f(x) = x^{\frac{2}{3}}(5 - x)$, $[-1, 8]$.

Solution: Note that $f(x) = 5x^{\frac{2}{3}} - x^{\frac{5}{3}}$, hence

$$f'(x) = \frac{10}{3x^{\frac{1}{3}}} - \frac{5x^{\frac{2}{3}}}{3} = \frac{10 - 5x}{3x^{\frac{1}{3}}}$$

This implies the critical points are $x = 0$ and $x = 2$. Next we compute the values of f at these points and the boundary points:

$$f(-1) = 6, \quad f(0) = 0, \quad f(2) = 3 \cdot 2^{\frac{2}{3}} \quad \text{and} \quad f(8) = -12$$

Hence the minimum is -12 and the maximum is 6. □

3. A man is at point A on a bank of straight river, 3km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then then run to B If he can row at 8 km/h and run at 10km/h. Where should he land to reach B as soon as possible?

   Solution: This is the exact problem we did in quiz five. We need to fine the minimum of the function
$$T(x) = \frac{\sqrt{x^2+9}}{8} + \frac{8-x}{10}$$
   over the domain $0 \le x \le 8$. Find the derivative first:
$$T'(x) = \frac{x}{8\sqrt{x^2+9}} - \frac{1}{10}.$$
   Set $T'(x) = 0$, we find $10x = 8\sqrt{x^2+9}$. We cancel 2 and square both sides and obtain: $25x^2 = 16(x^2+9)$. Solve for x and we obtain: $9x^2 = 16 \cdot 9$ and $x = 4$. Next we find the values of $T(0) = 3/8 + 8/10 = 1.175$, $T(8) = \sqrt{73}/8$ and $T(4) = 1.025$. And $T(4)$ is the smallest.

   Note; you can also use the angle $\theta = \angle ADC$ as the independent variable. Then the time function becomes
$$T(\theta) = \frac{3\csc\theta}{8} + \frac{8-3\cot\theta}{10}.$$
   Then the domain of this function is $\alpha \le \theta \le \frac{\pi}{2}$ where α is the angle such that $\tan\alpha = \frac{3}{8}$. Then we differentiate with respect to θ and obtain
$$T'(\theta) = -\frac{3\csc\theta\cot\theta}{8} + \frac{3\csc^2\theta}{10}.$$
   Set $T'(\theta) = 0$ and obtain $\cos\theta = \frac{4}{5}$. This is exactly $x = 4$.

4. A trapezoid is inscribed in a circle of radius 2. The longer side of the trapezoid coincides with a diameter of the circle. What is the maximum possible area of such a trapezoid.

   Solution: This is a problem from the homework and I have done the problem in class. Suppose the height of the trapezoid is h, then $h^2+x^2 = 4$ and this implies $h = \sqrt{4-x^2}$. And the area is
$$S(x) = \frac{1}{2}h(4+2x) = \sqrt{4-x^2}(2+x), \quad 0 \le x \le 1.$$
   Differentiate the function we find
$$\frac{dS}{dx} = -\frac{x(2+x)}{\sqrt{4-x^2}} + \sqrt{4-x^2} = \frac{4-2x-2x^2}{\sqrt{4-x^2}}.$$
   Then $\frac{dS}{dx} = 0$ implies that $4-2x-2x^2 = 0$. $2-x-x^2 = (1-x)(2+x)$. Hence $x = 1$ and $x = -2$. $x = -1$ is not in the domain. Find the value $S(0) = 4$, $S(1) = 3\sqrt{3}$ and $S(2) = 0$. Hence the maximum value is $S(1) = 3\sqrt{3}$. □