Mathematics 3100 - Homework II
Wednesday, February 11, 2004

1. Let $\{a_n\}$ and $\{b_n\}$ be increasing; are the following increasing? Prove or counterexample. (i) $\{a_n + b_n\}$ (ii) $\{a_n - b_n\}$.

   Solution: $\{a_n + b_n\}$ is increasing since

$$a_n \le a_{n+1} \quad \text{and} \quad b_n \le b_{n+1} \quad \text{implies} \quad a_n + b_n \le a_{n+1} + b_{n+1}.$$

   $\{a_n - b_n\}$ is not increasing since we can not take the difference of two inequalities. Take $a_n = n$ and $b_n = 2n$, then a_n and b_n are increasing but $a_n - b_n = -n$ is decreasing. □

2. Give an upper estimate for $\ln 3 = \int_1^3 \frac{dx}{x}$ as in the example 22C, by using (a) one trapezoid; (b) two trapezoids.

   Solution: (a). One trapezoid has area $\ln 3 \le 2 \times \frac{1 + \frac{1}{3}}{2} = \frac{4}{3}$.

   (b). Two trapezoid has area

$$1 \times \frac{1 + \frac{1}{2}}{2} + 1 \times \frac{\frac{1}{2} + \frac{1}{3}}{2} = \frac{7}{6}.$$

□

3. For each sequence, tell if it is bounded above or not

   (a) $a_n = \frac{1}{1} + \frac{1}{4} + \frac{1}{7} + \cdots + \frac{1}{3n+1}$.

   (b) $a_n = 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots + \frac{1}{n\sqrt{n}}$.

   Solution: (a). Compare a_n with the integral $\int_1^{n+1} \frac{dx}{1+3x}$, we have

$$a_n > \int_1^{n+1} \frac{dx}{1+3x} = \frac{1}{3}(\ln(3n+4) - \ln 7)$$

   Since $\ln(3n+4)$ is not bounded as $n \to \infty$. Hence a_n is not bounded above.

   (b).Compare a_n with the integral $\int_1^n \frac{dx}{x^{3/2}}$. we have

$$b_n < 1 + \int_1^n \frac{1}{x^{3/2}} = 1 + \frac{1}{2}(1 - \frac{1}{\sqrt{n}}) < \frac{3}{2}.$$

   Hence b_n is bounded above. □

4. If $|a_1 \sin b + a_2 \sin 2b + \cdots + a_n \sin nb| > n$, prove that $a_i| > 1$ for at least one of the a_i.

Proof. we can prove it by contraposition. Assume that $|a_i| \le 1$ for all $i = 1, 2, \cdots, n$. Then

$$\begin{aligned}|a_1 \sin b + a_2 \sin 2b + \cdots + a_n \sin nb| &\le |a_1||\sin b| + |a_2||\sin 2b| + \cdots + |a_n||\sin nb| \\ &\le |a_1| + |a_2| + \cdots + |a_n| \quad (\text{since } |\sin x| \le 1) \\ &\le n \quad (\text{since} \quad |a_i| \le 1 \quad \text{for all} \quad i)\end{aligned}$$

This contradicts to the statement that $|a_1 \sin b + a_2 \sin 2b + \cdots + a_n \sin nb| > n$. □

5. Give upper and lower estimates in terms of n along for $\left|\cos na + \frac{\sin nb}{n}\right|$.

Solution: We just apply the triangle inequality and $|\sin x| \le 1$ and $|\cos x| \le 1$.

$$\left|\cos na + \frac{\sin nb}{n}\right| \le |\cos na| + \left|\frac{\sin nb}{n}\right| \le 1 + \frac{1}{n}.$$

On the other hand

$$\left|\cos na + \frac{\sin nb}{n}\right| \ge \left||\cos na| - \left|\frac{\sin nb}{n}\right|\right| \ge \frac{1}{n}.$$

□

6. If $a \underset{\epsilon}{\approx} 1$ and $a \underset{\epsilon}{\approx} 2$, then $\epsilon > \frac{1}{2}$.

Proof. since $|a - 1| < \epsilon$ and $|a - 2| < \epsilon$,

$$1 = |1 - a + a - 2| \le |a - 1| + |a - 2| < 2\epsilon, \quad \epsilon > \frac{1}{2}.$$

□

7. Prove that if $a > 0$, the sequence $\frac{a^n}{n!}$ is monotone for large n.

Proof. The sequence involves the power and factorial, hence we consider the quotient $\frac{a_{n+1}}{a_n}$. This gives

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{(n+1)!} \times \frac{n!}{a^n} = \frac{a}{n+1} < 1$$

if $n + 1 > a$ or $n > a - 1$. Hence the sequence is decreasing if $n > a - 1$. □

8. Let $\{a_n\}$ be a sequence. $b_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$.

   (a) Prove that if $\{a_n\}$ is increasing, then $\{b_n\}$ is increasing.

   (b) Prove that if $\{a_n\}$ is bounded above, then $\{b_n\}$ is bounded above

Proof. (a). Since a_n is increasing, $a_k \le a_{n+1}$ for all $k < n+1$,

$$b_{n+1} = \frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} < \frac{a_{n+1} + a_{n+1} + \cdots + a_{n+1}}{n+1} = a_{n+1}.$$

and

$$(n+1)b_{n+1} - nb_n = (a_1 + a_2 + \cdots + a_{n+1}) - (a_1 + a_2 + \cdots + a_n) = a_{n+1}$$

This implies

$$n(b_{n+1} - b_n) = a_{n+1} - b_{n+1} > 0.$$

(b)If M is an upper bound for $\{a_n\}$, then $a_n \le M$ for all $n \in \mathbb{N}$. This implies

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \le \frac{M + M + \cdots + M}{n} = M.$$

Hence M is also an upper bound for $\{b_n\}$. □

9. The arithmetic-geometric mean inequality $\sqrt{ab} \le \frac{a+b}{2}$ for any $a, b \ge 0$.

Proof. This can be shown by the positivity of square.

$$0 \le (\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b \quad \text{so} \quad 2\sqrt{ab} \le a + b, \quad \sqrt{ab} \le \frac{a+b}{2}.$$

□

The figure illustrates the above inequality if we take h to be the height. Then the similarity of two right triangles gives $\frac{x}{a} = \frac{b}{x}$. Hence $x = \sqrt{ab}$ which is less than the radius $\frac{a+b}{2}$.