Mathematics 3100 - Homework III
Wednesday, February 11, 2004

1. Show that $\lim_{n\to\infty} \frac{2n-1}{n+2} = 2$ directly from the definition of limit.

   Proof. Given $\epsilon > 0$,

$$\left|\frac{2n-1}{n+2} - 2\right| = \left|\frac{2n-1-2(n+2)}{n+2}\right| = \frac{5}{n+2} < \epsilon$$

   if $n > \frac{5}{\epsilon} - 2$. □

2. Prove the sequence $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ has a limit.

   Proof. You just need to prove that the sequence is monotone and bounded. First we will show it is bounded above.

$$\begin{aligned} a_n &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \\ &\le \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1} \\ &= \frac{n}{n+1} < 1 \end{aligned}$$

   It obvious that $a_n > 0$. Hence $0 < a_n < 1$ for all $n \in \mathbb{N}$.

   Next we will show the sequence is monotone by taking the difference of a_n and a_{n+1}. Note that

$$\begin{aligned} a_{n+1} &= \frac{1}{n+1+1} + \frac{1}{n+1+2} + \cdots + \frac{1}{n+1+n} + \frac{1}{n+1+n+1} \\ &= \frac{1}{n+2} + \frac{1}{2+3} + \cdots + \frac{1}{2n+1} + \frac{1}{2n+2} \\ a_n &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \end{aligned}$$

   This implies that

$$a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} = \frac{1}{(2n+1)(2n+2)} > 0$$

   for all $n \in \mathbb{N}$. Therefore $\{a_n\}$ is increasing and bounded. So by the completeness property, the sequence has a limit. □

   As for the part (b), we can not use the K-ϵ principle, since n is not a constant but increases and $n\epsilon > 1$ because $n > \frac{1}{\epsilon}$. Hence what it has been shown in part (b) is $0 < a_n < n\epsilon$ but $n\epsilon > 1$. So this does not implies a_n is small.

3. Prove that $\lim_{n\to\infty}\left(\frac{1}{n^2+1}+\frac{1}{n^2+2}+\cdots+\frac{1}{n^2+n}\right)=0.$

Proof. Note that $\frac{1}{n^2+k}<\frac{1}{n^2}$ since $n^2+k>n^2$ for all $k>0$. This implies

$$\left(\frac{1}{n^2+1}+\frac{1}{n^2+2}+\cdots+\frac{1}{n^2+n}\right)<\frac{n}{n^2}=\frac{1}{n}.$$

Given $\epsilon>0$, we have $0<a_n<\frac{1}{n}<\epsilon$ is $n>\epsilon$. This proves that

$$\lim_{n\to\infty}\left(\frac{1}{n^2+1}+\frac{1}{n^2+2}+\cdots+\frac{1}{n^2+n}\right)=0.$$

□

4. For each sequence $\{a_n\}$, tell whether or not $a_n\to\infty$; if so, prove it directly from the definition of infinite limit.
(a) $a_n=\frac{n^2}{n-1}$. (b) $a_n=n^2|\cos n\pi|$.

Solution: (a). Note that $a_n=\frac{n^2}{n-1}>\frac{n^2}{n}=n$, therefore, given $M>0$, $a_n>M$ if $n>M$. This implies $\lim_{n\to\infty}a_n=\infty$.

(b). One only needs to note that $|\cos n\pi|=1$ and $a_n=n^2$. Therefore, given $M>0$, $a_n>M$ if $n>\sqrt{M}$. This implies $\lim_{n\to\infty}a_n=\infty$. □

5. Prove that if $a>1$, then $\lim_{n\to\infty}\frac{a^n}{n}=\infty$.

Proof. Follow the hint, we have

$$\begin{aligned}\frac{a^n}{n}&=\frac{(1+k)^n}{n}=\frac{1}{n}\left[1+nk+\frac{n(n-1)}{2!}k^2+\frac{n(n-1)(n-2)}{3!}k^3+\cdots+k^n\right]\\&>\frac{1}{n}\frac{n(n-1)}{2!}k^2=\frac{n-1}{2}k^2.\end{aligned}$$

since every term is positive and we just choose the third term. Then, given $M>0$, we have

$$\frac{a^n}{n}>M\quad\text{if}\quad\frac{n-1}{2}k^2>M\quad\text{this implies } n>\frac{2M}{k^2}+1.$$

Then by the definition, we have $\lim_{n\to\infty}\frac{a^n}{n}=\infty$. □

6. Prove that $na^n\to 0$ if $0<a<1$, using the result in the preceding exercise.

Proof. Since $0<a<1$, $r=\frac{1}{a}>1$. Then the preceding exercise implies $\frac{r^n}{n}\to\infty$, this yields that given $\epsilon>0$,

$$\frac{r^n}{n}>\frac{1}{\epsilon},\quad\text{for}\quad n\gg 1.$$

This implies $0<na^n<\epsilon$ for $n\gg 1$. Hence $\lim_{n\to\infty}na^n=0$. □

7. Prove that $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$, if $a > 0$.

Proof. Follows the hint, one considers the case of $a > 1$ first, let $a^{1/n} = 1 + e_n$. Since $a > 1$, $e_n > 0$. Then we can arise both sides to the power n and get

$$a = (1 + e_n)^n = 1 + ne_n + \frac{n(n-1)}{2!}e_n^2 + \cdots + e_n^n.$$

This implies

$$a > ne_n \quad \text{this implies} \quad 0 < e_n < \frac{a}{n}.$$

Given $\epsilon > 0$, $0 < e_n < \epsilon$ if $\dfrac{a}{n} < \epsilon$. This implies that $0 < e_n < \epsilon$ if $n > \dfrac{a}{\epsilon}$. Hence $e_n \to 0$. There $a^{1/n} \to 1$.

For the case of $0 < a < 1$, we let $r = \frac{1}{a} > 1$, then $a^{1/n} = \frac{1}{r^{1/n}}$. Since $r > 1$, we have $r^{1/n} > 1$ and given $\epsilon > 0$, $0 < r^{\frac{1}{n}} - 1 < \epsilon$ for $n \gg 1$. Then

$$|a^{1/n} - 1| = |\frac{1}{r^{1/n}} - 1| = \frac{r^{1/n} - 1}{r^{1/n}} << r^{1/n} - 1 < \epsilon \quad \text{for} \quad n \ggg 1.$$

Hence this shows $a^{1/n} \to 1$ for $0 < a < 1$. ☐

8. Prove $\lim_{n\to\infty} \int_2^3 \ln^n x dx = \infty$.

Proof. We need a lower estimate for the integral that is large. Since $\ln x$ is an increasing function which has the value 1 at $x = e$, we can find a point A between e and 3 (for example, we may take $A = \frac{e+3}{2}$). Then $B = \ln A > 1$ and $\ln x > B$ for $A < x < 3$. Since $\lim_{n\to\infty} B^n = \infty$, the definite integral must tends to ∞ also:

$$\text{given} \quad M > 0, \quad \int_2^3 \ln^n x dx \geq \int_A^3 \ln^n x dx \geq \int_A^3 B^n dx = B^n(3-A) > M, \quad \text{for } n \text{ large.}$$

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9. Show that $\lim_{n\to\infty} \int_0^1 (1 - x^2)^n dx = 0$.

Proof. This problem is can solved by following the example 3.7. Given $\epsilon > 0$, we show

$$\text{the area under } (1 - x^2)^n \text{ and over } [0, 1] < 2\epsilon, \quad \text{for } n \gg 1.$$

Divide up the area under the curve into two pieces as in the example: the first part is the area over interval $[0, \epsilon]$ and the second part is the area over interval $[\epsilon, 1]$.

$$\text{right-hand area} < \text{area of right hand rectangle} = \epsilon;$$
$$\text{left-hand area} < \text{area of right hand rectangle} = (1 - \epsilon)(1 - \epsilon^2)^n < \epsilon, \text{ for } n \gg 1,$$

since $0 < (1 - \epsilon^2) < 1$ (cf Theorem 3.4). Therefore

$$\int_0^1 (1 - x^2)^n dx < 2\epsilon \quad \text{for } n \gg 1.$$

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10. Let $\{a_n\}$ be a sequence and $\{b_n\}$ be its sequence of averages:

$$b_n = \frac{a_1 + a_2 + \cdots + b_n}{n}.$$

(a) Prove that if $a_n \to 0$, then $b_n \to 0$.

(b) Deduce from (a) that if $a_n \to L$, then also $b_n \to L$.

Proof. (a) Given $\epsilon > 0$, $a_n \to 0$ implies there exists N_1 such $|a_n| < \epsilon$ for all $n > N_1$. Then

$$\begin{aligned} |b_n| &= \left| \frac{a_1 + a_2 + \cdots + b_n}{n} \right| \\ &\le \frac{|a_1| + \cdots + |a_{N_1}| + |a_{N_1+1}| + \cdots + |a_n|}{n} \\ &< \frac{|a_1| + \cdots + |a_{N_1}|}{n} + \frac{n - N_1}{n}\epsilon \\ &< \frac{|a_1| + \cdots + |a_{N_1}|}{n} + \epsilon \end{aligned}$$

Since $\frac{1}{n} \to 0$, there exists N_2 such that

$$0 < \frac{|a_1| + \cdots + |a_{N_1}|}{n} < \epsilon \quad \text{for all } n > N_2.$$

Then we have $|b_n| < 2\epsilon$ for $n \gg 1$.

(b) Let $e_n = a_n - L$, then $e_n \to 0$ since $a_n \to L$. Then the average of $\{e_n\}$ tends to 0 also. But the average of e_n is $b_n - L$. Hence $b_n - L$ tends to 0. Therefore $b_n \to L$. □

11. Bernouill's inequality

$$(1 + h)^n \ge 1 + nh, \quad \text{for } h > 0.$$

Proof. The approach given here is the method of mathematical induction. For $n = 1$, $(1 + h)^1 = 1 + h$. Assume

$$(1 + h)^n \ge 1 + nh.$$

Then

$$(1+h)^{n+1} = (1+h)^n(1+h) \ge (1+nh)(1+h) = 1 + (n+1)h + nh^2 > 1 + (n+1)h$$

This prove the case for $n + 1$. Then the mathematical induction implies that this is true for all $n \in \mathbb{N}$. This above is true for $(1 + h) > 0$ or $h > -1$. □

12. Prove that a convergent sequence $\{a_n\}$ is bounded.

Proof. Since the sequence $\{a_n\}$ converges and assume that the limit is L, given $\epsilon = 1$, there exists N such that

$$|a_n - L| < 1 \quad \text{for } n > N$$

Apply the triangle inequality and we have $|a_n| - |L| < |a_n - L| < 1$, $|a_n| < |L| + 1$ for $n > N$. Next we let $M = \max\{|a_1|, \cdots, |a_N|, |L| + 1\}$, then we have $|a_n| < M$ for all $n \in \mathbb{N}$ and $\{a_n\}$ is bounded. □