# Mathematics 3100 - Homework VII
# Wednesday, April 7, 2004

1. Show that the following series can be rewritten as telescoping series, and use this to prove they converge and evaluate their sum.

$$(a) \quad \sum_{n=2}^{\infty} \frac{1}{n^2-1} \qquad (b) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+2)} \qquad (c) \sum_{n=1}^{\infty} \frac{1}{n(n+k)}, \ k \text{ integer} > 0.$$

*Proof.* When a series is written in the form

$$\sum_{0}^{\infty} a_n = s_0 + (s_1 - s_0) + (s_2 - s_1) + \cdots + (s_n - s_{n-1}) + \cdots .$$

and which makes self-evident the value of its $n$-th partial sum $s_n$, it is often called a *telescoping series.*

(a). We can first factor $n^2 - 1 = (n-1)(n+1)$ and apply the partial fraction to get

$$\begin{aligned}
\sum_{n=2}^{k} \frac{1}{n-1)(n+1)} &= \frac{1}{2} \sum_{n=2}^{k} \left( \frac{1}{n-1} - \frac{1}{n+1} \right) \\
&= \frac{1}{2} \left[ \sum_{n=2}^{k} \frac{1}{n-1} - \sum_{n=2}^{k} \frac{1}{n+1} \right] \\
&= \frac{1}{2} \left[ \sum_{n=1}^{k-1} \frac{1}{n} - \sum_{n=3}^{k+1} \frac{1}{n} \right] \\
&= \frac{1}{2} \left[ 1 + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1} \right]
\end{aligned}$$

Note that in the second line, the common part of the two sums is canceled. Since

$$\lim_{k \to \infty} \frac{1}{k} = \lim_{k \to \infty} \frac{1}{k+1} = 0,$$

this yields

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}.$$

(b). This can be worked just like (a). We first note that

$$\frac{1}{n(n+2)} = \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right).$$

We consider the partial sum

$$\begin{aligned}
s_k = \sum_{n=1}^{k} \frac{(-1)^{n-1}}{n(n+2)} &= \frac{1}{2} \sum_{n=1}^{k} \left( \frac{(-1)^{n-1}}{n} - \frac{(-1)^{n-1}}{n+2} \right) \\
&= \frac{1}{2} \left[ \sum_{n=1}^{k} \frac{(-1)^{n-1}}{n} - \sum_{n=3}^{k+2} \frac{(-1)^{n-3}}{n} \right] \\
&= \frac{1}{2} \left[ \sum_{n=1}^{k} \frac{(-1)^{n-1}}{n} - \sum_{n=3}^{k+2} \frac{(-1)^{n-1}}{n} \right] \\
&= \frac{1}{2} \left[ 1 - \frac{1}{2} - \frac{(-1)^k}{k+1} - \frac{(-1)^{k+1}}{k+2} \right]
\end{aligned}$$

This will yield that

$$\lim_{k\to\infty} s_k = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+2)} = \frac{1}{4}.$$

(c). Note that (a) is the special case of $k = 2$.

$$\begin{aligned} s_m &= \sum_{n=1}^{m} \frac{1}{n(n+k)} = \frac{1}{k}\sum_{n=1}^{m}\left(\frac{1}{n} - \frac{1}{n+k}\right) \\ &= \frac{1}{k}\left[\sum_{n=1}^{m}\frac{1}{n} - \sum_{n=1}^{m}\frac{1}{n+k}\right] \\ &= \frac{1}{k}\left[\sum_{n=1}^{m}\frac{1}{n} - \sum_{n=k+1}^{m+k}\frac{1}{n}\right] \\ &= \frac{1}{k}\left[\sum_{n=1}^{k}\frac{1}{n} - \sum_{n=m+1}^{m+k}\frac{1}{n}\right] \end{aligned}$$

Note that $\displaystyle\lim_{m\to\infty}\sum_{n=m+1}^{m+k}\frac{1}{n} = 0$. Hence

$$\sum_{n=1}^{\infty}\frac{1}{n(n+k)} = \frac{1}{k}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right).$$

$\square$

2. Given that $\displaystyle\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, evaluate $\displaystyle\sum_{0}^{\infty}\frac{1}{(2n+1)^2}$; cite theorems used.

*Solution:* We can write

$$\sum_{1}^{\infty}\frac{1}{n^2} = \sum_{1}^{\infty}\frac{1}{(2n)^2} + \sum_{0}^{\infty}\frac{1}{(2n+1)^2}$$

by splitting the even and odd terms. The sum of the even terms is

$$\sum_{1}^{\infty}\frac{1}{(2n)^2} = \frac{1}{4}\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{24}.$$

By the linearity theorem, we have

$$\sum_{0}^{\infty}\frac{1}{(2n+1)^2} = \sum_{1}^{\infty}\frac{1}{n^2} - \sum_{1}^{\infty}\frac{1}{(2n)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$$

$\square$

3. Prove that if $a_n \geq 0$ and $\sum a_n$ converges, then $\sum a_n^2$ converges.

*Proof.* Since $\sum a_n$ converges and $a_n \geq 0$, Theorem 7.2A implies $\displaystyle\lim_{n\to\infty} a_n = 0$. This implies that there exists $N$ such that $0 \leq a_n < 1$ for $n \geq N$. Hence $a_n^2 \leq a_n$ for $n \geq N$. This yields that

$$\sum_{N}^{\infty} a_n^2 < \sum_{N}^{\infty} a_n.$$

Then the convergence of $\sum a_n$ implies $\sum_N^\infty a_n$ is convergent by tail-convergence theorem. The comparison theorem for positive series implies the convergence of $\sum_N^\infty a_n^2$. Finally, apply the tail-convergence theorem again and obtain the convergence of $\sum a_n^2$. ☐

4. Generalize the preceding exercise: let $\sum a_n$ and $\sum b_n$ be two convergent series with non-negative terms. Prove that $\sum a_n b_n$ converges. Do this two ways:
   (a) prove and use an inequality relating the partial sums of the three series.
   (b) use the comparison series.

   *Proof.* (a). Note that the convergence of $\sum b_n$ implies that $b_n \to 0$ as $n \to \infty$. Then we apply the result that a convergent sequence is a bounded sequence. This yields that there exists $M > 0$ such that $0 \le b_n \le M$ for all $n$. Let $s_n = \sum_0^n a_n$ and $s'_n = \sum_0^n a_n b_n$. Then the boundedness of $b_n$ yields that $0 \le a_n b_n \le M a_n$. Hence we have

$$s'_n \le M s_n.$$

   Since $\{s_n\}$ is convergent and increasing, let $s_n \to A$, this implies

$$0 \le s'_n \le M \cdot A \quad \text{for all} \quad n.$$

   Since $\{s'_n\}$ is increasing and bounded, the completeness property implies $\{s'_n\}$ converges.

   (b). We can apply the previous exercise to obtain the convergence of $\sum a_n^2$ and $\sum b_n^2$. This implies $\sum (a_n^2 + b_n^2)/2$ converges. Note that $0 \le a_n b_n \le (a_n^2 + b_n^2)/2$. Then the comparison theorem implies $\sum a_n b_n$ converges. ☐

5. Prove that if $a_n$ and $b_n$ are non-negative, and $\sum a_n^2$ and $\sum b_n^2$ converge, then $\sum a_n b_n$ converges. Is this a stronger or weaker theorem than the one in the preceding exercise; why?

   *Proof.* This can be shown by the same method as (b) in the previous problem. The convergence of $\sum a_n^2$ and $\sum b_n^2$ imply the convergence of $\sum (a_n^2 + b_n^2)/2$ by the linearity theorem. Since

$$0 \le (a_n - b_n)^2 = a_n^2 + b_n^2 - 2a_n b_n \quad \text{implies} \quad 0 \le a_n b_n \le \frac{a_n^2 + b_n^2}{2}.$$

   the comparison theorem yields the convergence of $\sum a_n b_n$. ☐

6. Let $\sum a_n$ be a convergent series having the sum $S$. Let $\sum b_k$ be a new series, when the terms are formed by grouping the terms of $\sum a_n$ in pairs, and adding them. That is

$$b_1 = a_0 + a_1, \quad b_1 = a_2 + a_3, \cdots, \quad b_n = a_{2n} + b_{2n+1}, \cdots.$$

   Prove that $\sum b_k$ also converges, and to the same limit $S$.

*Proof.* The convergence of the infinite series $\sum a_n$ implies that the partial sum $s_k = \sum_0^k a_n$ converges and its limit is $S$. Then the partial sum

$$s'_k = \sum_0^k b_k = \sum_0^k (a_{2k} + a_{2k+1}) = s_{2k+1}$$

is a subsequence of $s_k$. The subsequence theorem implies that $s'_k$ converges to the same limit $S$. $\square$

7. Prove: if $\sum a_n$ is absolutely convergent and $\{b_n\}$ is bounded, then $\sum a_n b_n$ is convergent.

*Proof.* Since $\sum a_n$ is absolutely convergent, $\sum |a_n|$ converges. $\{b_n\}$ is bounded implies that there exists some constant $B$ such that $|b_n| \leq B$. This implies $|a_n b_n| \leq B|a_n|$. The comparison theorem and the convergence of $\sum |a_n|$ yield the convergence of $\sum |a_n b_n|$. Then the absolute convergence theorem implies the convergence of $\sum a_n b_n$. $\square$

8. (a) Prove: $\sum a_n$ absolutely converges $\Rightarrow \sum a_n^2$ converges.
   (b) Show that "absolutely" can not be dropped in part (a).

*Proof.* $\sum a_n$ absolutely converges implies the convergence of $\sum |a_n|$. Then the result 7.2/2 implies the convergence of $\sum |a_n|^2$. But $\sum |a_n|^2 = \sum a_n^2$, hence $\sum a_n^2$ converges. We will give an example to show that we can not drop the absolute convergence condition. Take $a_n = \dfrac{(-1)^{n-1}}{\sqrt{n}}$, then $\sum a_n$ converges but $\sum a_n^2 = \sum \frac{1}{n}$ diverges. $\square$

9. Prove that a conditionally convergent series has an infinity of positive terms and an infinity of negative terms.

*Proof.* Assume that $\sum a_n$ is conditionally convergent. Then $\sum |a_n|$ diverges. We will prove this by contradiction. Suppose that the series has finite positive terms or finite negative terms.

First we handle the case with finite positive terms. If $\sum a_n$ has finite positive terms, then there exists $N$ such that $a_n \leq 0$ for $n \geq N$.

$$\sum_{n=0}^{\infty} a_n = \sum_0^{N-1} a_n + \sum_{n=N}^{\infty} a_n \quad \Rightarrow \quad -\sum_{n=0}^{\infty} a_n = -\sum_0^{N-1} a_n - \sum_{n=N}^{\infty} a_n = -\sum_0^{N-1} a_n + \sum_{n=N}^{\infty} |a_n|.$$

This implies that $\sum_{n=N}^{\infty} |a_n|$ converges. The tail-convergence theorem implies the convergence of $\sum |a_n|$. This contradicts the divergence of $\sum |a_n|$.

If the the series has finite negative terms. Then there exists $N$ such that $a_n \geq 0$ for $n \geq N$.

$$\sum_{n=0}^{\infty} a_n = \sum_0^{N-1} a_n + \sum_{n=N}^{\infty} a_n = \sum_0^{N-1} a_n + \sum_{n=N}^{\infty} |a_n|.$$

This implies that $\sum_{n=N}^{\infty} |a_n|$ converges. The tail-convergence theorem implies the convergence of $\sum |a_n|$. This contradicates the divergence of $\sum |a_n|$. $\square$