Mathematics 3100 - Homework IX
Friday, April 24, 2004

1. Find the radius of convergence for the following power series.

Solution:

(a) $\sum_1^\infty \frac{x^n}{2^n\sqrt{n}}$. We can use either the root test and ratio test to find the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_n}{2_{n+1}}\right| = \lim_{n\to\infty}\frac{2^{n+1}\sqrt{n+1}}{2^n\sqrt{n}} = \lim_{n\to\infty} 2\sqrt{1+\frac{1}{n}} = 2.$$

So the ratio test gives the radius of convergence is 2.

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty}(2^n n^{\frac{1}{2}})^{\frac{1}{n}} = \lim_{n\to\infty} 2n^{\frac{1}{2n}} = 2,$$

where we use the fact that $n^{1/2n} \to 1$.

(b) $\sum_1^\infty \frac{(n!)^2}{(2n)!}x^n$. Apply the ratio test to find

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{(n!)^2}{(2n)!}\cdot\frac{(2n+2)!}{((n+1)!)^2} = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2} = 4.$$

The ratio test yields that the radius of convergence is 4.

(c) $\sum_1^\infty \frac{x^n}{\sqrt[n]{n}}$. We can use the ratio test to find the radius of convergence is

$$R = \lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}} = 1.$$

(d) $\sum_0^\infty \frac{(-1)^n x^{2n}}{4^n (n!)^2}$. Since this problem involves the factorial, we use the ratio test.

$$R = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{4^{n+1}((n+1)!)^2}{4^n(n!)^2} = \lim_{n\to\infty} 4(n+1)^2 = \infty.$$

This implies the power series converges for any $x \in \mathbb{R}$.

(e) $\sum_1^\infty \left(\frac{n+2}{n}\right)^n x^n$. Apply the root test. We first find

$$R = \lim_{n\to\infty} a_n^{-\frac{1}{n}} = \lim_{n\to\infty}\frac{n+2}{n} = 1.$$

(f) $\sum_2^\infty \frac{x^n}{(\ln n)^n}$. Apply the root test to find that

$$R = \lim_{n\to\infty}|a_n|^{-\frac{1}{n}} = \lim_{n\to\infty}\ln n = \infty.$$

Hence the series converges for any $x \in \mathbb{R}$.

(g) $\sum_{1} \frac{n^n x^n}{n!}$. Apply the ratio test.

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{1}{e}.$$

(h) $\sum_{1}^{\infty} \frac{n! x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$. We use the ratio test. Since

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)} \cdot \frac{1\cdot 3\cdot 5\cdots(2n+1)}{(n+1)!} = \lim_{n\to\infty} \frac{2n+1}{n+1} = 2.$$

□

2. For each series in Exercise 8.1/1 above, determine whether it converges at the end points $\pm\mathbb{R}$ of the interval of convergence.

   Solution:

   (a) $\sum_{1}^{\infty} \frac{x^n}{2^n\sqrt{n}}$. We have found the radius of convergence is 2. Take $x = 2$ we obtain the series $\sum 1/\sqrt{n}$ which is divergent. Take $x = 2$, we have the series $\sum(-1)^n/\sqrt{n}$ which is convergent by the Cauchy's test for alternating series.

   (b) $\sum_{1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$. The radius of convergence is 4. Set $x = 4$, we have the series $\sum \frac{4^n (n!)^2}{(2n)!}$. We can not use the ratio or root test to show it converges or diverges. The only method we can use is comparison test.

$$\begin{aligned} a_n = \frac{4^n(n!)^2}{(2n)!} &= \frac{(2^n n!)^2}{(2n)!} = \frac{(2\cdot 4\cdot 4\cdots 2n)^2}{1\cdot 2\cdot 3\cdots 2n} \\ &= \frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n-1)} \\ &= \frac{2}{1}\cdot\frac{4}{3}\cdots\frac{2n}{2n-1} > 2 \end{aligned}$$

   Hence $\lim a_n \neq 0$. This implies $\sum a_n$ diverges. Similar the power series diverges at $x = -4$.

   (c) $\sum_{1}^{\infty} \frac{x^n}{\sqrt[n]{n}}$. We find the radius of convergence is 1. At $x = \pm 1$, $|a_n| = \frac{1}{\sqrt[n]{n}}$. Hence $\lim a_n = 1 \neq 0$. Therefore the power series diverges at $x = \pm 1$.

   (d) $\sum_{0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}$. Since the power series converges for any $x \in \mathbb{R}$. There is no $\pm R$.

   (e) $\sum_{1}^{\infty} \left(\frac{n+2}{n}\right)^n x^n$. When $x = \pm 1$, we have

$$\lim_{n\to\infty} \left(\frac{n+2}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = e^2.$$

   Hence $\lim a_n \neq 0$, and the series divergence.

(f) $\sum_2^\infty \frac{x^n}{(\ln n)^n}$. Apply the root test to find that

$$R = \lim_{n\to\infty} |a_n|^{-\frac{1}{n}} = \lim_{n\to\infty} \ln n = \infty.$$

Hence the series converges for any $x \in \mathbb{R}$. We don't need to verify the convergence at $x = \pm R$.

(g) $\sum_1 \frac{n^n x^n}{n!}$. We have find the radius of convergence is $\frac{1}{e}$. We need to estimate $\frac{n^n}{n!e^n}$. This is not easy. You need to know that

$$(n-1)! \le \frac{n^n}{e^n} e \le n! \quad \Rightarrow \quad \frac{1}{ne} \le \left(\frac{n}{e}\right)^n \frac{1}{n!} \le 1.$$

Then by comparison test, we know $\sum \frac{n^n}{n!e^n}$ diverges. As for the case $x = -\frac{1}{e}$, we can apply Cauchy's test for alternating series to show it converges.

(h) $\sum_1^\infty \frac{n!x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$. This is the same problem as (b).

□

3. From the power series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots = \ln(1+x),$$

obtain the power series whose sum is $\ln\left(\frac{1+x}{1-x}\right)$.

Solution: Note that

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x).$$

From the power series for $\ln(1+x)$, we can obtain the power series for $\ln(1-x)$:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \cdots.$$

Hence we have

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \cdots + \frac{2x^{2n+1}}{2n+1} + \cdots.$$

□

4. What sum would Abel summation assign to $1 - 2 + 3 - 4 + 5 - \cdots$.

Solution: From the definition of the Abel summation, we need to compute

$$1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n.$$

Note that the right hand side is the derivative of the series $\sum(-1)^n x^{n+1} = x/(1+x)$. But

$$\frac{d}{dx}\frac{x}{1+x} = \frac{1}{(1+x)^2}.$$

Hence

$$\sum_{n=0}^{\infty}(-1)^n(n+1)x^n = \frac{1}{(1+x)^2}.$$

The Abel sum of $1-2+3-4+5-\cdots = \frac{1}{4}$. ☐

5. Let $\sum_0^\infty a_n$be a series and s_n its sequence of partial sums. Suppose $\sum a_n x^n$ converges for $|x|<1$. Let $f(x)$ be its sum. Then

$$\sum_{n=0}^{\infty} s_n x^n = \frac{f(x)}{1-x}, \quad \text{for} \quad |x|<1.$$

   (a) Illustrate the truth of the above for the series (i) $\sum 1$. (ii)$\sum 1/2^n$.

   (b) Prove the formula.

   Proof. this is just a application of the product formula of the power series. ☐

6. Determine with proof the radius of convergence of $\sum(\sin n)x^n$.

   Proof. First we note that $\sum \sin n$ diverges since $\lim \sin n$ does not exist. Hence the radius of convergence $R \le 1$. Next note that for any $|x|<1$, $|(\sin n)x^n| < |x|^n$ since $|\sin n| \le 1$. Since $\sum |x|^n$ conveges for $|x|<1$, $\sum |(\sin n)x^n|$ converges for $|x|<1$. Therefore the radius of convergence is 1. ☐