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Mathematics 3100 - Test One
Friday, March 26, 2004

1. Assume the geometric formula

$$1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

   (a) Derive a formula

$$1 + u^2 + u^4 + \cdots + u^{2n} = \frac{1}{1 - u^2} - e_n(u).$$

   and find $e_n(u)$ explicitly.

   (b) By integrating from 0 to $\frac{1}{2}$ of the above for both sides to show that

$$\frac{1}{2} + \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} + \cdots + \frac{1}{(2n+1) \cdot 2^{2n+1}} = \frac{1}{2} \ln 3 - \int_0^{1/2} e_n(u) du.$$

   (c) Show that $\lim_{n \to \infty} \int_0^{1/2} e_n(u) du = 0.$

Solution: (a). Let $a = u^2$ in the geometric series formula and we will have

$$1 + u^2 + u^4 + \cdots + u^{2n} = \frac{1 - u^{2n+2}}{1 - u^2} = \frac{1}{1 - u^2} - \frac{u^{2n+2}}{1 - u^2}.$$

Hence we have $e_n(u) = \frac{u^{2n+2}}{1 - u^2}$. (b). Integrate both sides from 0 to 1/2, we have

$$\begin{aligned}\frac{1}{2} + \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} + \cdots + \frac{1}{(2n+1) \cdot 2^{2n+1}} &= \frac{1}{2} \ln \left(\frac{1+u}{1-u} \right) \Bigg|_0^{1/2} - \int_0^{1/2} \frac{u^{2n+2}}{1 - u^2} du \\ &= \frac{1}{2} \ln 3 - \int_0^{1/2} \frac{u^{2n+2}}{1 - u^2} du.\end{aligned}$$

(c). Since $1 - u^2 > 1 - (1/2)^2 = 3/4$ for $0 \le u \le 1/2$,

$$|e_n| = \int_0^{1/2} \frac{u^{2n+2}}{1 - u^2} du \le \int_0^{1/2} \frac{4}{3} u^{2n+2} du = \frac{1}{3(2n+3)2^{2n+1}}.$$

This proves that $\lim_{n \to \infty} e_n = 0$ and

$$\lim_{n \to \infty} \left(\frac{1}{2} + \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} + \cdots + \frac{1}{(2n+1) \cdot 2^{2n+1}} \right) = \frac{1}{2} \ln 3.$$

□

2. Prove that $\lim_{n\to\infty}(n^2-n)^{\frac{1}{n}} = 1$ by using the error form principle 4.1 and the binomial expansion:

$$(1+k)^n = 1 + nk + \frac{n(n-1)}{2!}k^2 + \frac{n(n-1)(n-2)}{3!}k^3 + \cdots + k^n$$

Hint: $e_n + 1 = (n^2-n)^{\frac{1}{n}}$

Proof. First note that $(n^2-n)^{\frac{1}{n}} \geq 1$ for $n \geq 2$. Otherwise if $(n^2-n)^{\frac{1}{n}} < 1$ then $n^2 - n = n(n-1) < 1$. This is imposible for $n \geq 2$. Hence $e_n = (n^2-n)^{1/n} - 1 > 0$ for all $n > 1$. Then we write $(n^2-n)^{1/n} = 1 + e_n$ and arise to the n-th power both sides and get

$$n^2 - n = (1+e_n)^n = 1 + ne_n + \frac{n(n-1)}{2!}e_n^2 + \frac{n(n-1)(n-2)}{3!}e_n^3 + \cdots + e_n^n.$$

This implies

$$n^2 - n > \frac{n(n-1)(n-2)}{6}e_n^3 \quad \text{this implies} \quad e_n^3 < \frac{6}{n-2} \quad \text{and} \quad 0 < e_n < \sqrt[3]{\frac{2}{n-1}}.$$

Given $\epsilon > 0$, $0 < e_n < \epsilon$ if $\sqrt[3]{\dfrac{6}{n-1}} < \epsilon$. This implies that $0 < e_n < \epsilon$ if $n > 1 + \dfrac{6}{\epsilon^3}$.

□

3. Let $a_n = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$. Prove that $\lim_{n\to\infty} \frac{a_n}{n^{\frac{3}{2}}} = \frac{2}{3}$.

Proof. This can be proved by comparing $a_n = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$ with the area under the curve $y = \sqrt{x}$ over the interval $[0, n]$ and $[1, n+1]$. You should draw the picture and see a_n is the total area of the rectangles over interval $[0, n]$ with height $\sqrt{k}$ and base $[k-1, k]$. Since the curved area is inside the rectangles, we have

$$a_n \geq \int_0^n \sqrt{x}dx = \frac{2}{3}n^{\frac{3}{2}}.$$

On the other hand,

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \leq \int_1^{n+1} \sqrt{x}dx \leq \int_0^{n+1} \sqrt{x}dx = \frac{2}{3}(n+1)^{\frac{3}{2}}.$$

Combine these two inequalities, we have

$$\frac{2}{3}n^{\frac{3}{2}} \leq a_n \leq \frac{2}{3}(n+1)^{\frac{3}{2}}.$$

Next divide the chain of inequalities by $n^{\frac{3}{2}}$, then use the Squeeze Theorem:

$$\frac{2}{3} \leq \frac{a_n}{n^{\frac{3}{2}}} \leq \frac{2}{3}\left(\frac{n+1}{n}\right)^{\frac{3}{2}}.$$

Since $1 \leq \left(1 + \dfrac{1}{n}\right)^{\frac{3}{2}} \leq \left(1 + \dfrac{1}{n}\right)^2 = (1 + \dfrac{2}{n} + \dfrac{1}{n^2})$, the sequeeze theorem yields

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{\frac{3}{2}} = 1, \quad \text{we have} \quad \lim_{n\to\infty}\frac{a_n}{n^{\frac{3}{2}}} = \frac{2}{3}.$$

□

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4. Define a sequence recursively by $a_{n+1} = \sqrt{3a_n}$, $a_0 = 1$.

   (a) Prove $\{a_n\}$ is monotone and bounded.

   (b) Part (a) shows the limit L of $\{a_n\}$ Exists; determine L, and prove it is the limit.

Proof. (a). The recursive formula $a_{n+1} = \sqrt{3a_n}$ implies that $a_n > 0$ for all n. There are many different methods to show this sequence is monotone and bounded. We will use the method of mathematical induction. We first consider the case $a_0 > 3$. If $a_0 > 3$, then $a_1 = \sqrt{3a_0} > \sqrt{3 \cdot 3} = 3$. Assume that $a_n > 3$, then $a_{n+1} = \sqrt{3a_n} > \sqrt{3 \cdot 3} = 3$. Hence we have $a_n > 3$ for all $n \in \mathbb{N}$ by the mathematical induction. Then

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{3a_n}}{a_n} = \sqrt{\frac{3}{a_n}} < 1 \quad \text{since} \quad a_n > 3.$$

Hence the sequence is decreasing and bounded from below by 3 if $a_0 > 3$.

Similarly, we can prove that if $0 < a_0 < 3$, the sequence is increasing and bounded above by 3.

(b). The completeness property implies that $\lim_{n\to\infty} a_n$ exists. Assume that the limit is L, then the recursive formula $a_{n+1} = \sqrt{3a_n}$ implies $L = \sqrt{3L}$. This yields $L = 0$ or $L = 3$. But part (a) implies $L \neq 0$. Hence $L = 3$. We will give two methods of showing that $L = 3$. We will just consider the case $a_0 \geq 3$. The other case can proved similarly. The first method use the error analysis. Let $e_n = a_n - 3$, Then $e_{n+1} + 3 = \sqrt{3(e_n + 3)} = \sqrt{3e_n + 9}$. Hence

$$e_{n+1} = \sqrt{3e_n + 9} - 3 = [\sqrt{3e_n + 9} - 3] \cdot \frac{\sqrt{3e_n + 9} + 3}{\sqrt{3e_n + 9} + 3} = \frac{3e_n}{\sqrt{3e_n + 9} + 3}.$$

Since $a_n > 3$, this gives $e_n = a_n - 3 > 0$. Hence

$$0 < e_{n+1} = \frac{3e_n}{\sqrt{3e_n + 9} + 3} \leq \frac{3e_n}{3 + 3} = \frac{e_n}{2}.$$

This yields

$$e_1 < \frac{e_0}{2}, \quad e_2 < \frac{e_1}{2} < \frac{e_0}{4}, \cdots, e_n < \frac{e_0}{2^n}.$$

Since $\frac{1}{2^n} \to 0$, the **Squeeze Theorem** implies $\lim e_n = 0$.

The second method is to find a formula for a_n in terms n and a_0. Note that $a_1 = \sqrt{3a_0} = 3^{\frac{1}{2}} a_0^{\frac{1}{2}}$. Then

$$a_2 = 3^{\frac{1}{2}} a_1^{\frac{1}{2}} = 3^{\frac{1}{2}+\frac{1}{4}} a_0^{\frac{1}{4}} \quad \text{and} \quad a_3 = 3^{\frac{1}{2}} a_2^{\frac{1}{2}} = 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} a_0^{\frac{1}{8}}.$$

We can prove inductively

$$a_n = 3^{\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}} a_0^{\frac{1}{2^n}}.$$

Next we note that

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \quad \text{by the geometric series formula.}$$

Hence we can write

$$a_n = 3^{1-\frac{1}{2^n}} a_0^{\frac{1}{2^n}} = 3(\frac{a_0}{3})^{\frac{1}{2^n}}.$$

Since $a^{\frac{1}{n}} \to 1$ for $a > 0$, $\lim a_n = 3\lim(\frac{a_0}{3})^{\frac{1}{2^n}} = 2$.

We can also find this formula by definition. $a_{n+1} = \sqrt{3a_n}$ implies

$$\frac{a_{n+1}}{3} = \sqrt{\frac{a_n}{3}} = \left(\frac{a_n}{3}\right)^{\frac{1}{2}} = \left(\left(\frac{a_{n-1}}{3}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = \cdots = \left(\frac{a_0}{3}\right)^{\frac{1}{2^{n+1}}}.$$

Hence we have $a_n = 3\left(\frac{a_0}{3}\right)^{\frac{1}{2^n}}$. Since $\lim_{n\to\infty}\left(\frac{a_0}{3}\right)^{\frac{1}{n}} = 1$ for $a_0 > 0$, $\lim_{n\to\infty}\left(\frac{a_0}{3}\right)^{\frac{1}{2^n}} = 1$ for $\left(\frac{a_0}{3}\right)^{\frac{1}{2^n}}$ is a subsequence of $\left(\frac{a_0}{3}\right)^{\frac{1}{n}}$. □

5. Let $x_0 = 0$ and $x_1 = 1$. Then continue the sequence by letting each new term by the average of the preceding two:

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}, \quad n \geq 2.$$

(a) Prove that $\{x_n\}$ is A Cauchy sequence.

(b) Find $\lim_{n\to\infty} x_n$. (Hint: $x_n = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_1 - x_0) + x_0$).

Proof. This is just the homework problem with a=0 and b=1. (a) Note that

$$x_n - x_{n-1} = \frac{x_{n-1} + x_{n-2}}{2} - x_{n-1} = -\frac{x_{n-1} - x_{n-2}}{2},$$

then we use the recursion and obtain

$$x_n - x_{n-1} = -\frac{x_{n-1} - x_{n-2}}{2} = \frac{x_{n-2} - x_{n-3}}{4} = \cdots$$
$$= (-\frac{1}{2})^{n-1}(x_1 - x_0) = (-\frac{1}{2})^{n-1}(b - a).$$

This implies

$$\begin{aligned}
|x_n - x_m| &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&= |b-a|\left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^m}\right) \\
&= \frac{|b-a|}{2^{n-1}}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{m-n+1}}\right) \\
&< \frac{|b-a|}{2^{n-2}} < \epsilon, \quad \text{for} \quad m > n \gg 1
\end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence.

(b). From (a), we can see that

$$x_n - x_{n-1} = (-\frac{1}{2})^{n-1}(b - a).$$

Follows the hints we have

$$
\begin{aligned}
x_n &= (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_1 - x_0) + x_0 \\
&= a + (b-a) - \frac{b-a}{2} + \frac{b-a}{4} - \frac{b-a}{8} + \cdots + (b-a)(-\frac{1}{2})^{n-1} \\
&= a + (b-a)\left(1 + (-\frac{1}{2}) + (-\frac{1}{2})^2 + \cdots + (-\frac{1}{2})^{n-1}\right) \\
&= a + (b-a)\frac{1-(-\frac{1}{2})^n}{1+\frac{1}{2}}
\end{aligned}
$$

Hence $\lim_{n\to\infty} x_n = a + \frac{2}{3}(b-a) = \frac{a+2b}{3}$. □

6. The sequence of *Fibonacci Fractions* is defined recursively by

$$a_{n+1} = \frac{1}{1+a_n}, \quad n \geq 1; \quad a_1 = 1.$$

(a) Prove that $\frac{1}{2} \leq a_n \leq 1$.

(b) Prove that $\{a_n\}$ is a Cauchy sequence.

Proof. this is the example 6.4. □