

Mathematics 3100 - Test Three
Monday, April 26, 2004

1. Show that the following series can be rewritten as telescoping series, use this to prove they converge and evaluate their sum.

$$(a) \quad \sum_1^{\infty} \frac{1}{n^2 - \frac{1}{4}}. \quad (b) \quad \sum_1^{\infty} \frac{(-1)^{n-1}}{n(n+2)}.$$

Solution: a). We can first factor $n^2 - 1/4 = (n - 1/2)(n + 1/2)$ and apply the partial fraction to get

$$\begin{aligned} \sum_{n=1}^k \frac{1}{(n - 1/2)(n + 1/2)} &= \sum_{n=2}^k \left(\frac{1}{n - 1/2} - \frac{1}{n + 1/2} \right) \\ &= \left[\sum_{n=1}^k \frac{1}{n - 1/2} - \sum_{n=1}^k \frac{1}{n + 1/2} \right] \\ &= \left[\sum_{n=1}^k \frac{1}{n - 1/2} - \sum_{n=2}^{k+1} \frac{1}{n - 1/2} \right] \\ &= \left[2 - \frac{1}{k + 1/2} \right] \end{aligned}$$

Note that in the second line, the common part of the two sums is canceled. Since $\lim_{k \rightarrow \infty} \frac{1}{k + 1/2} = 0$, this yields

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1/4} = 2.$$

(b). This can be worked just like (a). We first note that

$$\frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

We consider the partial sum

$$\begin{aligned} s_k &= \sum_{n=1}^k \frac{(-1)^{n-1}}{n(n+2)} = \frac{1}{2} \sum_{n=1}^k \left(\frac{(-1)^{n-1}}{n} - \frac{(-1)^{n-1}}{n+2} \right) \\ &= \frac{1}{2} \left[\sum_{n=1}^k \frac{(-1)^{n-1}}{n} - \sum_{n=3}^{k+2} \frac{(-1)^{n-3}}{n} \right] \\ &= \frac{1}{2} \left[\sum_{n=1}^k \frac{(-1)^{n-1}}{n} - \sum_{n=3}^{k+2} \frac{(-1)^{n-1}}{n} \right] \\ &= \frac{1}{2} \left[1 - \frac{1}{2} - \frac{(-1)^k}{k+1} - \frac{(-1)^{k+1}}{k+2} \right] \end{aligned}$$

This will yield that

$$\lim_{k \rightarrow \infty} s_k = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+2)} = \frac{1}{4}.$$

□

2. Given that $\sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, evaluate $\sum_0^{\infty} \frac{1}{(2n+1)^2}$;

Solution: We can write

$$\sum_1^{\infty} \frac{1}{n^2} = \sum_1^{\infty} \frac{1}{(2n)^2} + \sum_0^{\infty} \frac{1}{(2n+1)^2}$$

by splitting the even and odd terms. The sum of the even terms is

$$\sum_1^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}.$$

By the linearity theorem, we have

$$\sum_0^{\infty} \frac{1}{(2n+1)^2} = \sum_1^{\infty} \frac{1}{n^2} - \sum_1^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$$

□

3. Test each of the following series for convergence:

$$(a) \sum_1^{\infty} \frac{\sin n}{n^2}. \quad (b) \sum_2^{\infty} \frac{1}{n(\ln n)^p}. \quad (c) \sum \frac{(-1)^n}{n^{1/2}}. \quad (d) \sum_0^{\infty} \left(\frac{n}{n+1}\right)^{\frac{1}{2}n^2}.$$

Solution: (a). Since $|\sin n| \leq 1$, we have $\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$. $\sum 1/n^2$ converges, then the comparison theorem implies that the series absolutely converges, so it converges.

(b). First consider the case $p = 1$. Apply the integral test. You need first to show that $f(x) = (x \ln x)^{-1}$ is positive and decreasing. Since

$$\int_2^A \frac{1}{x \ln x} dx = \ln(\ln x)|_2^A = \ln(\ln A) - \ln(\ln 2) \rightarrow \infty, \quad \text{as } A \rightarrow \infty.$$

Hence the integral $\int_2^{\infty} \frac{1}{x \ln x} dx$ diverges. Therefore the series diverges. For the general case, first compare with the case of $p = 1$, since $n \ln n \geq n(\ln n)^p$ for $p \leq 1$ and $n \geq 3$. The divergence of $\sum (n \ln n)^{-1}$ implies the series $\sum [n(\ln n)^p]^{-1}$ diverges for $p \leq 1$. We apply the integral test to take care of the case $p > 1$. You need first to show that $f(x) = \frac{1}{x(\ln x)^p}$ is positive and decreasing. Since

$$\int_2^A \frac{1}{x(\ln x)^p} dx = \frac{1}{1-p} (\ln x)^{1-p} \Big|_2^A = \frac{1}{p-1} [(\ln 2)^{1-p} - (\ln A)^{1-p}] \rightarrow \frac{1}{(p-1)(\ln 2)^{p-1}}$$

as $A \rightarrow \infty$, if $p > 1$. Hence the integral $\int_2^{\infty} \frac{1}{x(\ln x)^p} dx$ converges for $p > 1$. Therefore the series converges for $p > 1$ and diverges if $p \leq 1$.

(c). Cauchy's test for alternating series implies the series converges. You just need to note that $\frac{1}{n^{1/2}} > 0$ and decreases, and approaches 0 as $n \rightarrow \infty$.

(d). Apply the root test. We first find

$$\lim_{n \rightarrow \infty} a_n^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^{\frac{n}{2}} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^{\frac{n}{2}}} = \frac{1}{\sqrt{e}} < 1.$$

Hence the series converges.

□

4. Find the radius of convergence of the following power series.

(a) $\sum_1^{\infty} \frac{(n!)^2}{(2n)!} x^n$. (b) $\sum_2^{\infty} \frac{n^{n^2} x^n}{(1+n)^{n^2}}$.

(c) For (a), determine whether it converges at the endpoint R of the interval of convergence.

Solution: (a). Apply the ratio test to find

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = \lim_{n \rightarrow \infty} \frac{(n!)^2}{(2n)!} \cdot \frac{(2n+2)!}{((n+1)!)^2} = \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4.$$

The ratio test yields that the radius of convergence is 4.

(b). Apply the root test. We first find

$$R = \lim_{n \rightarrow \infty} a_n^{-\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

(c). Set $x = 4$ in (a), we have the series $\sum \frac{4^n (n!)^2}{(2n)!}$. We can not use the ratio or root test to show it converges or diverges. The only method we can use is comparison test.

$$\begin{aligned} a_n &= \frac{4^n (n!)^2}{(2n)!} = \frac{(2^n n!)^2}{(2n)!} = \frac{(2 \cdot 4 \cdot 6 \cdots 2n)^2}{1 \cdot 2 \cdot 3 \cdots 2n} \\ &= \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)} \\ &= \frac{2}{1} \cdot \frac{4}{3} \cdots \frac{2n}{2n-1} > 2 \end{aligned}$$

Hence $\lim a_n \neq 0$. This implies $\sum a_n$ diverges. \square

5. Find the Abel Sum of the series $\sum (-3)^n = 1 - 3 + 9 - 27 + 81 - \cdots$.

Solution: Note that the series $\sum (-3)^n$ diverges since $\lim (-3)^n$ does not exist. Follow the definition of the Abel sum, we consider the power series $\sum_{n=0}^{\infty} (-3x)^n$. This is a geometric series and it is convergent for $|x| < 1/3$ and has limit $f(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$. Set $x = 1$, we obtain the Abel sum is $1/4$. \square

6. (**Bonus Problem**) Suppose that f is increasing on $[1, \infty)$.

(a) Show that $f(1) + f(2) + \cdots + f(n-1) < \int_1^n f(x) dx < f(2) + f(3) + \cdots + f(n)$.

(b) Choose $f(x) = \ln x$ and show that $\frac{n^n}{e^{n-1}} < n! < \frac{n^{n+1}}{e^{n-1}}$.

(c) Show that $\sum_{n=1}^{\infty} \frac{n^n}{n! e^n}$ divergence.

Solution: (a). This is very easy. Since $f(x)$ is increasing on $[1, \infty)$. we have

$$f(k) \leq f(x) \leq f(k+1), \quad \text{for } 1 \leq k \leq x \leq k+1 \Rightarrow f(k) \leq \int_k^{k+1} f(x) dx \leq f(k+1).$$

Sum up the above inequality from $k = 1$ to $k = n - 1$, we have

$$f(1) + f(2) + \cdots + f(n-1) < \int_1^n f(x) dx < f(2) + f(3) + \cdots + f(n).$$

This is what we need to prove. If you can draw the right picture, I will give you some points.

(b). Set $f(x) = \ln x$, we have

$$\ln(n-1)! \leq \int_1^n \ln x dx < \ln n! \quad \Rightarrow \quad \ln(n-1)! \leq \left. x \ln x - x \right|_1^n \leq \ln n!.$$

Next note that

$$\left. x \ln x - x \right|_1^n = n \ln n - n + 1.$$

This yields

$$(n-1)! < e^{n \ln n - n + 1} < n! \quad \Rightarrow \quad (n-1)! < \frac{n^n}{e^{n-1}} < n!.$$

We just rewrite the inequality to get

$$\frac{n^n}{e^{n-1}} < n! \quad \text{and} \quad (n-1)! < \frac{n^n}{e^{n-1}} \quad \Rightarrow \quad n! < \frac{n^{n+1}}{e^{n-1}}.$$

(c). This above inequalities imply that

$$e < \frac{n!e^n}{n^n} < ne \quad \Rightarrow \quad \frac{1}{ne} < \frac{n^n}{n!e^n} < \frac{1}{e}.$$

Since $\frac{1}{e} \sum 1/n$ diverges, the Comparison test implies that $\sum_{n=1}^{\infty} \frac{n^n}{n!e^n}$ diverges. \square