Mathematics 2250 - Sample Final Examinations - Spring 2008

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

$$\frac{d}{dx}a^x = (\ln a)a^x$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \tan x \sec x$$

$$\frac{d}{dx}\csc x = -\cot x \csc x$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

$$\lim_{x\to 0}\frac{\sin ax}{x} = a$$

PART A

Each of the following 18 multiple-choice questions has **exactly one** correct answer. 4 marks for a correct answer, 0 for no answer or a wrong answer or an unclear answer or indicating more than one answer. You are not required to justify your answer in this part.

1. $\lim_{x\to 1} \dfrac{2x^2 - x - 1}{x^2 + x - 2} =$
   (a) Undefined.
   (b) -1.
   (c) 1
   (d) $\frac{1}{2}$
   (e) 2

2. $\lim_{x\to 1} \dfrac{x^4 - 1}{x^3 - 1} =$
   (a) $\frac{3}{4}$
   (b) $\frac{4}{3}$
   (c) 0
   (d) -1
   (e) 1

3. $\lim_{x\to 0} \left(\dfrac{\sin 3x}{2x} + \dfrac{\sin 5x}{\sin 4x} + \dfrac{1 - \cos x}{x} \right) =$
   (a) $\frac{13}{4}$
   (b) $\frac{11}{4}$
   (c) $\frac{3}{2}$
   (d) $\frac{11}{2}$
   (e) $\frac{7}{4}$

4. $\lim_{x\to 0} \dfrac{\sqrt{1+3x^2}-\sqrt{1-3x^2}}{2x^2} =$
   (a) 0
   (b) 3
   (c) $\frac{3}{2}$
   (d) ∞
   (e) $\frac{1}{2}$

5. $\lim_{x\to\infty} \dfrac{\sqrt[3]{x^3-8x+1}}{4x-3} =$
   (a) 0
   (b) $\frac{2}{3}$
   (c) $-\frac{1}{3}$
   (d) $\frac{1}{4}$
   (e) $-\frac{1}{2}$

6. Let $f(x) = x + \dfrac{1}{x}$ on $[1,4]$. The Mean Value Theorem says that there must be some number c between 1 and 4 so that $f'(c) = \dfrac{f(4)-f(1)}{4-1}$. The number c must be
   (a) $\frac{16}{9}$
   (b) $\sqrt{\frac{3}{2}}$
   (c) 2
   (d) $\frac{5}{2}$
   (e) $\frac{9}{4}$

7. If $f(x) = \ln(1 + 3x + e^{2x})$, then $f'(0) =$

(a) 3
(b) $\frac{5}{2}$
(c) 2
(d) 1
(e) $\frac{1}{2}$

8. Let $f(x) = (1 + x)^3$. If $g(x) = f^{-1}(x)$, i.e., if g is the inverse of f, then $g'(8) =$

(a) $\frac{1}{12}$
(b) $\frac{1}{243}$
(c) $\frac{1}{8}$
(d) $\frac{1}{24}$
(e) $\frac{1}{128}$

9. The graph of $f(x) = 3x^{\frac{1}{2}} - x^{\frac{3}{2}}$ $(x \geq 0)$ has a horizontal tangent line at $x =$

(a) 3
(b) $\sqrt{3}$
(c) $\frac{1}{3}$
(d) 1
(e) 0

10. Let f and g be differentiable functions such that $f(1) = 2$, $f'(1) = \frac{1}{2}$, $g(2) = 3$ and $g'(2) = \frac{1}{3}$. Let $h(x) = f(g(f(x)))$. If $h'(1) = \frac{5}{2}$, What is the value of $f'(3)$?

(a) 15
(b) 12
(c) 9
(d) 6
(e) 10

11. Let $f(x) = (x^2 - x)e^{-x}$. Then the graph of f is concave down on the interval

(a) $(-\infty, 1)$
(b) $(-1, 2)$
(c) $(2, \infty)$
(d) $(1, 4)$
(e) $(4, \infty)$

12. The function $f(x) = x + \dfrac{1}{x}$ has

(a) a local min. at $x = 1$ and $x = -1$.
(b) a local max. at $x = 1$ and $x = -1$.
(c) a local min. at $x = 1$ and a local max. at $x = -1$.
(d) a local max. at $x = 1$ and a local min. at $x = -1$.
(e) no local max. and no a local min.

13. Let $f(x) = \dfrac{\ln x}{x^2}$, for $x > 0$. Then f has

(a) a point of inflection at $x = \sqrt{e}$
(b) a global max. at $x = e^2$
(c) a global min. at $x = e^2$
(d) a global max. at $x = \sqrt{e}$
(e) a global min. at $x = \sqrt{e}$

14. If $x^2 + xy^2 + y = 7$, then the value of $\dfrac{dy}{dx}$ at the point $(1, 2)$ must be

(a) $-\frac{6}{5}$
(b) -1
(c) -5
(d) 1
(e) 0

15. The sum of two positive numbers is 8, then the smallest possible value of the sum of their squares must be

(a) 34
(b) 26
(c) 30
(d) 32
(e) 28

16. A 13 ft. ladder is leaning against a vertical wall, and the bottom of the ladder begins to slide away from the wall. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 ft. from the wall, sliding at 2ft./sec.?

(a) $\frac{2}{3}$ ft./sec.
(b) $\frac{1}{2}$ ft./sec.
(c) $\frac{5}{6}$ ft./sec.
(a) 1 ft./sec.
(a) $\frac{1}{3}$ ft./sec.

17. Let $f(x) = \begin{cases} 2(1-x) & \text{if} x \le 1 \\ 3(x^2-1) & \text{if} x > 1. \end{cases}$ Then the average value of $f(x)$ over the interval $[0,3]$ equals

(a) 21 ft./sec.
(b) $\frac{11}{2}$ ft./sec.
(c) 7 ft./sec.
(a) -2 ft./sec.
(a) $\frac{11}{3}$ ft./sec.

18. If $f(x) = \displaystyle\int_{2x}^{2x^2} t^t dt$, then $f'(1) =$

(a) 0
(b) 8
(c) 10
(d) 16
(e) 12

PART B

Answer all questions in PART B in spaces provided. Show all your work. Any answer in PART B without justification may receive very little or no credit. Use the back of each page for rough work.

1.[15 points] Find the derivative of the function $f(x) = x^3 + \dfrac{1}{x}$ by the **definition of the derivative**. Find the equation of the line which is tangent to f and with **slope** equal to 2.
Hint: $x^3 - a^3 = (x - a)(x^2 + xa + a^2)$.

2.[15 points] Evaluate the indefinite integrals:

(a) $\int \sqrt{x}(1-x)^3 dx.$

(b) $\int (e^{2x} - e^{-x})^2 dx.$

3.[10 points] A water tank has the shape of an inverted circular cone with the base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min. Find the rate at which the water level is rising when the water is 2 m deep.

4.[10 points] A trapezoid is inscribed in a semicircle of radius 2. The longer of the two parallel sides of the trapezoid coincides with the diameter of the semicircle. What is the maximum possible area of such a trapezoid? Justify your conclusion by citing the theorem you have used.

5.[10 points] P is point on the positive x-axis and Q is a point on the positive y-axis. What is the shortest possible distance between P and Q if the line joining P and Q is tangent to the curve $y = \dfrac{3}{x}$ at some points? Justify your conclusion by citing the theorem you have used.

6.[10 points] A straight line passing through the point $(3, 4)$ intersects the positive x-axis at the point $(p, 0)$ and the positive y-axis at the point $(0, q)$. What is the minimum value of $p + q$? Justify your conclusion by citing the theorem you have used.

(Note: This is not an easy question and will marked very strictly)