Mathematics 2250 - Quiz One
Monday, January 14, 2008

1. Evaluate $\lim_{x\to 3} \dfrac{\sqrt{16+x^2}-5}{x-3}$.

   Solution: This problem is similar to Problem **42** of section **2.2**. I just changed the problem to test your skill in algebra.

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{16+x^2}-5}{x^2} &= \lim_{x\to 0} \frac{\sqrt{16+x^2}-5}{x-3} \cdot \frac{\sqrt{16+x^2}+5}{\sqrt{16+x^2}+5} \\
&= \lim_{x\to 3} \frac{(16+x^2)-25}{(x-3)(\sqrt{16+x^2}+5)} \\
&= \lim_{x\to 3} \frac{x^2-9}{(x-3)(\sqrt{16+x^2}+5)} \\
&= \lim_{x\to 3} \frac{x+3}{\sqrt{16+x^2}+5} \\
&= \frac{6}{\lim_{x\to 3}\sqrt{16+x^2}+\lim_{x\to 3} 5} \quad \text{(quotient and sum laws)} \\
&= \frac{3}{5} \quad \text{(root law)}
\end{aligned}$$

□

2. Evaluate $\lim_{x\to 0} \dfrac{1}{x}\left(\dfrac{1}{\sqrt{16+x}} - \dfrac{1}{4}\right)$

   Solution: This problem is very close to the example I have done in the class. (I changed 9 to 16 and 3 to 4)

$$\begin{aligned}
\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{\sqrt{16+x}} - \frac{1}{4}\right) &= \lim_{x\to 0} \frac{1}{x}\frac{4-\sqrt{16+x}}{4\sqrt{16+x}} \\
&= \lim_{x\to 0} \frac{4-\sqrt{16+x}}{4x\sqrt{16+x}} \cdot \frac{4+\sqrt{16+x}}{4+\sqrt{16+x}} \\
&= \lim_{x\to 0} \frac{16-(16+x)}{4x\sqrt{16+x}(4+\sqrt{16+x})} \\
&= \lim_{x\to 0} \frac{-x}{4x\sqrt{16+x}(4+\sqrt{16+x})} = \lim_{x\to 0} \frac{-1}{4\sqrt{16+x}(4+\sqrt{16+x})} \\
&= -\frac{1}{128} \quad \text{quotient, product and substitution laws.}
\end{aligned}$$

□

3. Compute the average rate of changes of the function $f(x) = x + \dfrac{4}{x}$ over the interval [1,x]: $m(x) = \dfrac{f(x)-f(1)}{x-1}$. Then find the limit $\lim_{x\to 1} m(x)$. You will **not** get any point by using the derivative.

Solution: We have done a similar problem in class. First we compute $f(x) - f(a)$ and factorize the factor $(x - a)$ from it.

$$\begin{aligned} f(x) - f(a) &= \left(x + \frac{4}{x}\right) - \left(a + \frac{4}{a}\right) \\ &= (x - a) + \left(\frac{4}{x} - \frac{4}{a}\right) \\ &= (x - a) + 4\frac{(a - x)}{xa} = (x - a)(1 - \frac{4}{xa}) \end{aligned}$$

This implies $\dfrac{f(x) - f(a)}{x - a} = 1 - \dfrac{4}{xa}$. Hence

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \left(1 - \frac{4}{xa}\right) = 1 - \frac{4}{a^2}.$$

When $a = 1$, $f(1) = 5$ and $m(1) = -3$.

If you work directly starting with $a = 1$, you will find that

$$\frac{f(x) - f(1)}{x - 1} = \frac{x^2 - 5x + 4}{x(x - 1)}.$$

The next question is how to factor out $(x - 1)$ from the numerator $x^2 - 5x + 4$. I suppose that everyone should know how to do it.

$$(x^2 - 5x + 4) = (x - 4)(x - 1).$$

Hence

$$\frac{f(x) - f(1)}{x - 1} = \frac{x^2 - 5x + 4}{x(x - 1)} = \frac{(x - 4)}{x},$$

and $\lim_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = -3..$ □