Mathematics 2250 - Quiz Three
Wednesday, February 6, 2008

1. Consider the function $f(x) = \dfrac{x}{1-2x}$.

   (1) Apply the definition of the derivative $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ to find $f'(a)$.

   (2) Apply the quotient rule to verify your solution.

   (3) Find the equation of the line tangent to $y = f(x)$ with slope equal to 1.

Solution: This is not a difficult problem. Follows the definition

$$\begin{aligned} f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \frac{1}{x-a} \cdot \left(\frac{x}{1-2x} - \frac{a}{1-2a} \right) \\ &= \frac{1}{x-a} \cdot \frac{x(1-2a) - a(1-2x)}{(1-2x)(1-2a)} \\ &= \frac{1}{x-a} \cdot \frac{x - 2xa - a + 2xa)}{(1-2x)(1-2a)} = \lim_{x \to a} \frac{1}{(1-2x)(1-2a)} = \frac{1}{(1-2a)^2} \end{aligned}$$

If you like the other definition of the derivative. The computation is more less the same.

$$\begin{aligned} f(x+h) - f(x) &= \frac{x+h}{1-2(x+h)} - \frac{x}{1-2x} \\ &= \frac{(x+h)(1-2x) - x(1-2x-2h)}{(1-2x-2h)(1-2x)} \\ &= \frac{x(1-2x) + h - 2xh - x(1-2x) + 2xh}{(1-2x-2h)(1-2x)} \\ &= \frac{h}{(1-2x-2h)(1-2x)} \end{aligned}$$

Hence

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{(1-2x-2h)(1-2x)} = \frac{1}{(1-2x)^2}$$

Next we apply the quotient rule to verify that

$$f'(x) = \frac{1 \cdot (1-2x) - x \cdot (-2)}{(1-2x)^2} = \frac{1 - 2x + 2x}{(1-2x)^2} = \frac{1}{(1-2x)^2}.$$

Set $f'(x) = 1$, we get $(1-2x)^2 = 1$ and This implies $1 - 2x = \pm 1$ and $x = 0$ or $x = 1$. So there are two points on the curve that the slope of the tangent line equals to 1 at these points. These points are $(0, 0)$ and $(1, -1)$. So the equations of the line tangent to the curve at $(0, 0)$ is $y = x$, and the equation of the line tangent to the curve at $(1, -1)$ is $y + 1 = x - 1$. □

2. Apply the angel difference formula for the sine, $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ and the definition of derivative $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ to the derivative of $f(x) = \tan x$.

Solution: I will not show how to find the derivative by the quotient role here.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h} = \lim_{h\to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x} \\ &= \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos(x+h)\cos x} \\ &= \lim_{h\to 0} \frac{\sin(h)}{h\cos(x+h)\cos x} = \frac{1}{\cos^2 x} \end{aligned}$$

□

3. A population of chipmunks moves into a new region at time $t = 0$, At time t (in month), the population numbers

$$P(t) = 20[10 + 3t + (0.4)t^2].$$

(a) How long does it take for the population to double its initial size?

(b) What is the rate of growth of the population when $P = 400$.

Solution: (a). We first find the initial population by setting $t = 0$ and getting $P(0) = 200$. We need to find $t > 0$ such that $P(t) = 400$. This implies

$$20[10 + 3t + (0.4)t^2] = 400, \quad 0.4t^2 + 3t + 10 = 20, \quad 0.4t^2 + 3t - 10 = 0.$$

You can find t either by factoring out the polynomial or using the formula.

$$0.4t^2 + 3t - 10 = (0.4t - 1)(t + 10) = 0 \quad \text{implies } t = 2.5 \quad \text{or} \quad t = -10 < 0$$

Since we need to find positive time, we have $t = 2.5$.

(b). We first find the rate of change of the population is

$$P'(t) = 60 + 16t \quad \text{then} \quad P'(2.5) = 100.$$

□