MATH 4010/6010 (Chastkofsky) TEST 1 Feb. 22, 2008

1. (25 points)

a. Define what is meant by a group.

Solution : A group G is a set G together with an operation : $G \times G \to G$ such that the operation is associatve, there is an identity, and every element has an inverse.

b. Prove that if G is a group and H is a non-empty finite subset of G which is closed under the operation of G, then H is a subgroup of G.

Solution :Let x be in H. Since H is closed under the operation, all powers of x belong to H. Since H is finite, there are integers m and n with $m > n$ such that $x^n = x^m$, so that $x^{m-n} = 1$. So 1 belongs to H and $x \cdot x^{m-n-1} = 1$, so x has an inverse. The operation is associative, so H is a group.

c. Give an example of a group G and an infinite subset H of G such that H is closed under the operation of G, but H is not a subgroup of G.

Solution : Let $G = \mathbb{Z}$ under addition, and $H = \mathbb{Z}^+$. The sum of 2 positive integers is a positive integer, so H is closed under addition, but negative numbers are not in H, so not every element in H has an inverse.

2. (25 points)

a. Prove that if G is a group such that every non-identity element has order 2 , then G is abelian.

Solution : The identity of G commutes with everything. Let a and b be any 2 non-identity elements of G. Then $(ab)^2 = 1$, so $abab = 1$, and $ba = a^{-1}b^{-1}$ but $a^{-1} = a$ and $b^{-1} = b$ so $ba = ab$.

b. Prove that every group of order 4 is abelian.

Solution : By Lagrange's theorem, the order of every element of the group divides 4, so ever non-identity element has order 2 or 4. If there is an element of order 4, the group is a cyclic, so abelian. If every element has order 2, it is abelian by part (b).

c. List all groups of order 4 (up to isomorphism). (Proof is not needed; list any one group from each isomorphism class).

Solution : $C_2, C2 \times C_2$

3. (25 points) Let $\mathbb{Z}_8$ be the group of integers modulo 8 under addition, and $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ be the group of ordered pairs (a, b),where a is in $\mathbb{Z}_4$ and b is in $\mathbb{Z}_2$, also under addition. Define $\phi : \mathbb{Z}_8 \to \mathbb{Z}_4 \oplus \mathbb{Z}_2$ by$\phi(\overline{x}) = (\overline{x}, \overline{x})$ (where x is in $\mathbb{Z}$). (Note We are using the same symbol $\overline{x}$ to represent elements in $\mathbb{Z}_8, \mathbb{Z}_4, \mathbb{Z}_2$ respectively.)

a. Prove that ϕ is a well-defined homomorphism.

Solution : If $\overline{x} = \overline{y}$ in Z_8 then 8 divides $x - y$, so both 4 and 2 divide $x - y$, so $\overline{x} = \overline{y}$ in Z_4 and in Z_2, so $(\overline{x}, \overline{x}) = (\overline{y}, \overline{y})$ in $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. Thus ϕ is well-defined. $\phi(\overline{x} + \overline{y}) = (\overline{x + y} = (\overline{x} + \overline{y}) = \phi(\overline{x}) + \phi(\overline{y})$, so ϕ is a homomorphism.

b. Find the kernel of ϕ and the image of ϕ. To what group that you listed in problem 2(c) is the image isomorphic?

Solution : We compute, (omitting bars), $\phi(0) = \phi(4) = (0, 0), \phi(1) = \phi(5) = (1, 1), \phi(2) = \phi(6) = (2, 0), \phi(3) = \phi(7) = (3, 1)$, so the kernel is $\{0, 4\}$ and the image is $\{((0, 0), (1, 1), (2, 0), (3, 1)\}$. The image is isomorphic to C_4 ((1,1) is a generator).

c. State the Fundamental Theorem of Homomorphisms of groups (the text's version will suffice here) and explain how the computation you did in part (b)agrees with this theorem.

Solution : The theorem says that if ϕ is a group homomorphism on G, then $G/Ker(\phi)$ is isomorphic ti $Im\phi$. In our case the quotient is a cyclic group of order 4 (generated by $\overline{1}$: the quotient of a cyclic group is cyclic) and so is the image.

4. (25 points) Let $\sigma = (123), \tau = (124), \pi = (1234)$, considered as elements of S_4.

a. Let H be the (cyclic) subgroup of S_4 generated by π. Write down the elements of H (in cycle notation). Is H a normal subgroup of S_4? Explain.

Solution : We compute $H = \{(1234), (13)(24), (4321) = (1432), Identity\}$. H is not normal because there are 4-cyles not in H (eg (1243)) and all 4-cycles are conjugate in S_4.

b. What is the index of H in S_4? Write down the elements of one other coset (of your choosing) besides H.

Solution : The index is 24/4=6. $(12)H = \{(234), (1324), (143), (12)\}$ eg.

c. Find all elements γ in S_4 such that $\gamma\sigma\gamma^{-1} = \tau$. Are any in A_4? Explain.

Solution : $\sigma = (123)(4) and \tau = (124)(3) = (412)(3) = (241)(s)$ so γ is either (3,4), (1432), or (1243). These are each cycles of even length, so are odd and not in A_n.

d. There is a normal subgroup of order 4 in S_4. Write down its elements. To which group in 2(c) is it isomorphic? Explain briefly how you know it is normal.

Solution : $N = \{Id., (12)(34), (13)(24), (14)(23)\}$. It is isomorphic to $C_2 \times C_2$. H has all elements consisting of two disjoint 2-cycles, and these form a conjugacy class, so whenever an element of H is conjugated, another element of H is the result.