MATH 4010/6010 (Chastkofsky) TEST 2 April 9, 2008

Justify your answers, and show all work.

1. (20 points)

(a) Define what is meant by the action of a group on a set.

Solution: A group G acts on a set S if there is a homomorphism $\phi : G \to Perm(s)$ (Alternatively, if $(gh).x = g.(h.x)$ and $1.x = x$ for all $g, h \in G$ and $x \in S$.)

(b) Suppose G acts on the set S and let x be an element of S. Define what is meant by the orbit of x under the action of G and the stablizer of x under the action.

Solution: The orbit of x is $\{g.x : g \in G\}$ and its stabilizer is $\{g \in G : g.x = x\}$

(c) Suppose G acts on the set S and let x be an element of S. Solution:Let H be the stablizer of x under the action. Prove that H is a subgroup of G and that the size of the orbit of x equals the index of H in G.

Solution:H is not empty since $1 \in H$. If $g, h \in H$ then $(gh).x = g.h.x = g.x = x$ and $g^{-1}.x = g^{-1}.g.x = 1.x = x$. Map $O_x \to$ cosets of H in G by $g.x \to gH$. We have $g.x = h.x \iff h^{-1}g \in H \iff gH = hH$, so this map is well-defined and 1-1. A given coset gH is mapped onto by $g.x$, so it is a bijection.

2. (20 points) (a) Let G be a group and H a subgroup of G. Let S be the set of left cosets of H in G. Let $\phi : G \to Perm(S)$ be defined by $\phi(g)(aH) = gaH$. Prove that the stabilizer of the left coset aH under this action is aHa^{-1}.

Solution:g is in the stabilizer iff $gaH = aH \iff a^{-1}gaH = h \iff a^{-1}ga \in H \iff g \in aHa^{-1}$

(b) Prove that the kernel of the map ϕ given in (a) is $\bigcap aHa^{-1}$, the intersection over all a in G.

Solution:g is in the the kernel of ϕ iff it is in the stabilizer of every coset $aH, a \in G$, so the result follows from part (a).

3. (20 points)

(a) Define what is meant by a Sylow-p subgroup of a group.

Solution: If p is a prime dividing the order of the group G, and if p^n divides $|G|$ but p^{n+1} does not, then a Sylow-p subgroup is one whose order is p^n.

(b) State the 3 Sylow theorems.

Solution: For any prime p dividing the order of a group, a Sylow-p subgroup always exists, any two are conjugate in G, and their number is congruent to 1 mod p.

4. (20 points) Let G be a group of order 24. Let H be the Sylow-2 subgroup of G and let $K = \bigcap aHa^{-1}$, the intersection over all a in G. Explain briefly how we know that K must be a normal subgroup of G. Prove that either $K = H$ or K is a subgroup of H of index 2. In each of the possible 2 cases, find the order of K and identify G/K. (Hint: use problem 2).

Solution: K is the kernel of the map ϕ defined in problem 2, so is a normal subgroup. K is a subgroup of H, so $|K|$ divides 8. By the Fundamental Theorem of Homomorphisms of Groups, $G/K \cong Im(\phi)$ which is isomorphic to a subgroup of S_3 (there are 3 cosets of H in G) , so $|G|/|K|$ divides 6, so $|G|/6 = 4$ divides $|K|$. So the order of K is 4 or 8. In the first case, G/K is (isomorphic to) a subgroup of S_3 of order 6, so is isomorphic to S_3. In the second case, G/K is (isomorphic to) a subgroup of S_3 of order 3, so is isomorphic to C_3.

5. (20 points) (a) Prove that there is no simple group of order 56.

Solution: The number of Sylow-7 subgroups is congruent to 1 mod 7, and equals the index of the normalizer of a Sylow-7 in the group, so divides 8. So the number is 1 or 8. If it is 1, there is 1 Sylow-7, so it must be normal (by Sylow II). If there are 8, there are 6x8=48 elements of order 7, since any 2 groups of order 7 intersect trivially. The remaining elements must all belong to the Sylow-2, so there is only 1 Sylow-2, which must be normal as before.

(b) How many Sylow-7 subgroups are there in S_7? Justify your answer.

Solution:Elements of order 7 are 7-cycles. There are 7! ways of putting 1...7 in a 7-cycle, but each 7-cycle can be represented in 7 such ways, so there are 7!/7=6! 7-cycles. Each Sylow-7 contains 6 different 7-cycles and any 2 groups of order 7 intersect trivially, so there are 6!/6=120 Sylow-7's.