MATH 8100 MIDTERM II - FALL 2007

Problem 1.

a) Prove that the set of irrational numbers form a dense G_δ set.

b) For any positive integer $\lambda > 0$, prove that there exits a dense G_δ set $E \subseteq \mathbb{R}$ such that: $m(E) = \lambda$.

c) (Extra Credit) Prove that the set of rational numbers is not a G_δ set.

Problem 2. Construct a sequence of measurable functions: $f_n : [0,1] \to \mathbb{R}$ such that $f_n \to 0$ in measure, but $f_n(x) \not\to 0$ for every $x \in [0,1]$.

Recall that, a sequence of measurable functions f_n converge to a function f is measure, if for every $\varepsilon > 0$:

$$m(\{x : \ |f_n(x) - f(x)| > \varepsilon\}) \to 0 \quad \text{as} \quad n \to \infty$$

Hint: Try $f_n = \chi_{E_n}$ for an appropriate sequence of sets $E_n \subseteq [0,1]$.

Problem 3. Let $f : [0,1] \to \mathbb{R}$ be continuous. Prove that:

$$\lim_{n \to \infty} (n+1) \int_0^1 x^n f(x)\, dx \; = \; f(1)$$

Problem 4. Let H be a Hilbert space, $x \in H$ and $\{x_n\}_{n=0}^{\infty} \subset H$. Assume that $\|x_n\| \to \|x\|$ and for every $y \in H$: $(x_n, y) \to (x, y)$ as $n \to \infty$.

Prove that $\|x_n - x\| \to 0$ as $n \to \infty$.

Problem 5. Let $\{f_n\}_{n=0}^{\infty}$ be an orthonormal set in $L^2(E)$ with $|E| < \infty$. Assume that there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all x and n.

i) Show that $\sum_{n=1}^{\infty} n^{-1} f_n$ converges to some $f \in L^2(E)$ in the L^2 norm.

ii) Let $S_N = \sum_{n=1}^{N} n^{-1} f_n$ and $g_N = \sum_{n=1}^{N} |S_{(n+1)^2} - S_{n^2}|$. Prove that g_N is a Cauchy sequence in $L^2(E)$.

i3) Proceed as in the proof of the completeness of L^p spaces, to show that $\lim_{N\to\infty} S_{N^2}(x) = f(x)$ for a.e. x, and finally to show that

$\lim_{N\to\infty} S_N(x) = f(x)$ for a.e. x as well that is the sum in part i) converges point-wise almost everywhere.

Problem 6. Let $F : [0,1] \times [0,1] \to [0,1]$ be a measurable function, such that for each $x \in [0,1]$

$$\int_0^1 F(x,y)\, dy \geq 1/2$$

Show that the measure of the set

$$E = \{y \in [0,1] : \int_0^1 F(x,y)\, dx \geq 1/6\}$$

is at least $1/3$.

Hint: Use Fubini's theorem.