EXERCISES/PROBLEMS.

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I'll list below exercises and problems and sometimes even open problems. Many of the problems are in fact lead to known theorems, which we have not time to discuss in class, so I try to "decompose them" into problems (often with hints), to give a guide to their proofs. None of these have to be turned in, but the best way to understand the material is really to try to some of them.

1. Combinatorics.

Problem 1. A triple of integers: x, y, z is called *Schur triple*, if $x + y = z$. Let $r \in \mathbb{N}$. Find an $N \in \mathbb{N}$, such that if $[1, N]$ is r-colored, then there is a monochromatic Schur triple.

Hint: Let $V = [1, N]$ be the vertex set of a graph, and color the edge (m, n) with the color of $m - n$. Use Ramsey's theorem.

Problem 2. Show that there are at least $N^3/24$ monochromatic triangle in in every 2-coloring of the edges of a graph with N vertices.

Hint: Estimate the number of two-colored triangles from above.

ii) Show that if $[1, N]$ is two colored then there are at least $N^2/48$ monochromatic Schur triple in it.

Problem 3. Let $R(3, m)$ denote the smallest positive integer N, such that if the edges of a graph on N vertices are colored by m colors, then there must be a monochromatic triangle.

i) Prove that

$$R(3, m+1) - 1 \; \leq \; (m+1)(R(3, m) - 1) + 1$$

ii) Use this to show: $R(3, m) \leq e\, m! + 1$ where e denotes the Euler number.

Hint: Use the fact that: $1 + \frac{1}{1!} + \ldots + \frac{1}{m!} \leq e$ for all m.

Problem 4*. (Erdös-Szekeres) Let $k \in \mathbb{N}$. Show that $\exists N \in \mathbb{N}$ such that any N points on the plane contain k points which form a convex polygon.

i) Show that $N = 5$ is good for $k = 4$.

ii) Color a 4-tuple of points Red if they form a convex polygon, and Blue otherwise. Can you formulate and prove a generalization of Ramsey's theorem, which would imply that there exists k points such that any 4-tuple chosen from them is Red?

Why would this solve the problem?

Problem 5. Let G be a vertex regular bipartite graph, with vertex set $V = V_1 \cup V_2$, such that $|V_1| = |V_2| = N$ and $N_x = \delta N$ for all $x \in V$. Let $0 < \varepsilon < \delta$ be fixed.

i) Show that if V is ε-regular then

$$||N_x \cap N_y| - \delta^2 N| \leq \frac{\varepsilon}{2} N$$

for all but at most $\frac{\varepsilon}{2} N^2$ pairs of vertices (x, y).

ii) For the converse, assume that: $||N_x \cap N_y| - \delta^2 N| \leq \varepsilon N$ for all but at most εN^2 pairs of vertices (x, y); and show that the number of 4-cycles in G is bounded by $(\delta^4 + \varepsilon_1) N^4$ with $\varepsilon_1 = 3\varepsilon/2 + \varepsilon^2$.

i3) Can you formulate (and prove) the analogues statement for sets $A \subseteq \mathbb{Z}/N$ of density δ? Use the correspondence $A \to G_A$ between sets and bipartite graphs described in class.

Problem 6. Let G be a bipartite graph with vertex set $V = V_1 \cup V_2$, such that $|V_1| = |V_2| = N$. Suppose V has $|E| = \delta N^2$ edges and is ε-regular with $0 < \varepsilon \leq \delta/3$.

Show that there are at least: $(\delta - 2\varepsilon)^{2k} N^{2k}$ $2k$-cycles in G for all $k > 1$. A $2k$-cycle is a set of vertices (not necessarily distinct): $m_1, m_2, \ldots, m_{2k}$, $m_{2k+1} = m_1$, such that $m_{2i-1} \in V_1$, $m_{2i} \in V_2$ and $(m_i, m_{i+1}) \in E$ for all $1 \leq i \leq 2k$.