

6120 Take-Home Final Exam, Due Monday, May 5. Drop your exam off in my mailbox by noon.

Instructions: You may use your book, your notes and homework, and the supplementary notes on the course webpage, but no other sources. Please contact me if you think there is an error or if a question needs clarification.

1. i) Let X be an $n \times n$ matrix, and let A be an invertible $n \times n$ matrix. Prove that $\det(A + tX) = \det(A)(1 + t \operatorname{tr}(A^{-1}X) + o(t))$, where tr stands for *trace*, the sum of the diagonal entries. (Suggestions: First prove it for $A = I$. Think of X as a list of columns.)
- ii) Let $Sl(n)$ denote the set of $n \times n$ matrices A such that $\det(A) = 1$. Prove that $Sl(n)$ is a manifold.

2. Let $Q \subset \mathbb{R}^n$ be a closed rectifiable subset. Let $\mathcal{A} = \{R_1, \dots, R_k\}$ be a collection of subsets of Q , such that each R_i is rectifiable, $R_i \cap R_j$ has measure zero for $i \neq j$, and $\cup_i R_i = Q$.

- i) Draw a picture of the situation.
- ii) Let $f : Q \rightarrow \mathbb{R}$ be a continuous function. Define

$$U(f, \mathcal{A}) = \sum_i \sup_{x \in R_i} (f(x))v(R_i)$$

$$L(f, \mathcal{A}) = \sum_i \inf_{x \in R_i} (f(x))v(R_i)$$

Prove that $L(f, \mathcal{A}) \leq \int_Q f \leq U(f, \mathcal{A})$

- iii) Define $\operatorname{diam}(R_i) = \sup_{x, y \in R_i} |x - y|$, and define $\mu(\mathcal{A}) = \max_i (\operatorname{diam}(R_i))$. Prove that for all $\epsilon > 0$ there exists $\delta > 0$ such that for all \mathcal{A} , if $\mu(\mathcal{A}) < \delta$, then $U(f, \mathcal{A}) - L(f, \mathcal{A}) < \epsilon$.

3. Let $M \subset \mathbb{R}^n$ be a manifold of dimension k . Prove that for all $p \in M$ there is a set $\{i_1, i_2, \dots, i_k\} \subset \{1, \dots, n\}$ such that x_{i_1}, \dots, x_{i_k} restricts to a local coordinate system on a neighborhood of p in M .

4. Let $f : [a, b] \rightarrow \mathbb{R}$ be a smooth function, where $0 < a < b$. Let S be the surface of revolution obtained by revolving the graph of f around the y -axis. In calculus II one learns that the surface area of S is given by the formula

$$A = \int_a^b 2\pi x \sqrt{1 + f'(x)^2} dx .$$

Derive this formula, starting with the observation that S is a smooth manifold.

5. One knows that if $M \subset \mathbb{R}^n$ is a smooth manifold, then in a neighborhood of each point there is a local defining function. This raises the question of whether

there is always a global defining function. The following result shows that if M is nonorientable, then there is no global defining function.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ be a smooth function. Let $M = f^{-1}(0)$. Assume that $Df(p)$ has rank $n - k$ at every point $p \in M$. (Thus M is a k -dimensional manifold.) Prove that M is orientable. (Hint: Write $f = (f_1, \dots, f_{n-k})$. Let $\alpha : U \rightarrow M$ be a patch, let $x \in U$ and let $p = \alpha(x)$. What do you observe about the list of vectors $\nabla f_1(p), \dots, \nabla f_{n-k}(p), D_1(\alpha)(x), \dots, D_k(\alpha)(x)$?)

6. Let $p_1, \dots, p_s \in \mathbb{R}^2$ be a set of points in the plane. Let ω be a closed one-form on $\mathbb{R}^2 - \{p_1, \dots, p_s\}$. Let $M \subset \mathbb{R}^2$ be a compact manifold of dimension 2, such that for all i , $p_i \notin \partial M$. Show that there are constants $\delta_1, \dots, \delta_s$, independent of M , such that $\int_{\partial M} \omega = \sum_{\{i \mid p_i \in M\}} \delta_i$.

7. Definition: Let $X \subset Y \subset \mathbb{R}^n$. A *retraction* from $Y \rightarrow X$ is a continuous function $f : Y \rightarrow X$ that restricts to the identity function on X .

i) Let $M \subset \mathbb{R}^2$ denote the closed unit disk, with the origin removed. Write down a retraction from M to ∂M .

Let M be a compact oriented n -dimensional manifold with nonempty boundary. A basic result in topology is that there is no smooth retraction from M to its boundary. (The preceding part of this problem shows that compactness is a necessary hypothesis.) Complete the following proof of this theorem.

ii) Prove that if N is a nonempty compact manifold of dimension $n - 1$, then there is a smooth $(n - 1)$ -form $\omega \in \Omega^{n-1}(N)$ such that $\int_N \omega \neq 0$.

iii) To get a contradiction, let $f : M \rightarrow \partial M$ be a retraction. By part ii, choose an $(n - 1)$ -form $\omega \in \Omega^{n-1}(\partial M)$ such that $\int_{\partial M} \omega \neq 0$. Then apply Stokes's theorem to $f^*(\omega)$.

8. (Maxwell's equations) Let $U \subset \mathbb{R}^3$ be an open set. Smooth functions $F : U \times \mathbb{R} \rightarrow \mathbb{R}^3$ are called *time-dependent vector fields*. Let $E = (E_1, E_2, E_3)$ and $B = (B_1, B_2, B_3)$ be time-dependent vector fields. Taking (x, y, z) as coordinates on U and t as the coordinate on \mathbb{R} , let

$$\omega_{(E,B)} = E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt + B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy .$$

i) Express the equation $d\omega_{(E,B)} = 0$ as two equations involving the classical operators *div* and *curl*.

ii) Two of the four equations of Maxwell state that $d\omega_{(E,B)} = 0$, where E is the electric field and B is the magnetic field. Prove that if U is contractible, then there is a time-dependent vector field $A(x, y, z, t)$ (the *vector potential*) and a scalar function $\phi(x, y, z, t)$ (the *scalar potential*) such that

$$\begin{aligned} E &= -\frac{\partial A}{\partial t} - \text{grad}(\phi) \\ B &= \text{curl}(A) \end{aligned}$$

