

DIFFERENTIATION UNDER THE INTEGRAL SIGN

Theorem 1. *Let $Q \subset \mathbb{R}^n$ be a rectangle. Let $(a, b) \subset \mathbb{R}$ be an open interval, and let $f : Q \times (a, b) \rightarrow \mathbb{R}$ be a function such that for all $t \in (a, b)$, $f(x, t)$ is an integrable function of $x \in Q$. Let*

$$\phi(t) = \int_{x \in Q} f(x, t) .$$

Assume that $\frac{\partial f}{\partial t}$ is defined and continuous for all $(x, t) \in Q \times (a, b)$. Then ϕ is differentiable, and

$$\frac{d\phi}{dt}(t) = \int_{x \in Q} \frac{\partial f}{\partial t}(x, t) .$$

Proof. Fix $t \in (a, b)$. Let $\epsilon > 0$. A continuous function on a compact set is uniformly continuous, so there exists a $\delta > 0$ such that for all $x, y, \in Q$ and all $h \in [-\delta, \delta]$,

$$\left| \frac{\partial f}{\partial t}(y, t+h) - \frac{\partial f}{\partial t}(x, t) \right| < \epsilon .$$

For all $x \in Q$ and h such that $0 < |h| < \delta$, there exists by the Mean Value Theorem a number $c_{x,h}$ with $|c_{x,h}| < |h|$, such that

$$\frac{1}{h}(f(x, t+h) - f(x, t)) = \frac{\partial f}{\partial t}(x, t + c_{x,h}) .$$

Then

$$\begin{aligned} & \left| \frac{\phi(t+h) - \phi(t)}{h} - \int_{x \in Q} \frac{\partial f}{\partial t}(x, t) \right| \\ &= \left| \int_{x \in Q} \left(\frac{1}{h}(f(x, t+h) - f(x, t)) - \frac{\partial f}{\partial t}(x, t) \right) \right| \\ &\leq \int_{x \in Q} \left| \frac{1}{h}(f(x, t+h) - f(x, t)) - \frac{\partial f}{\partial t}(x, t) \right| \\ &= \int_{x \in Q} \left| \frac{\partial f}{\partial t}(x, t + c_{x,h}) - \frac{\partial f}{\partial t}(x, t) \right| \leq \epsilon v(Q) . \end{aligned} \tag{0.1}$$

□