THE EXTERIOR ALGEBRA

1. The determinant

The determinant is a preferred element of $\mathcal{A}^n(\mathbb{R}^n)$. If $T \in \mathcal{A}^n(\mathbb{R}^n)$, then the input to T is a sequence $(v_1, ..., v_n)$ where each $v_i \in \mathbb{R}^n$. It is convenient, however, to regard $(v_1, ..., v_n)$ as an $n \times n$ matrix. In this way of thinking, elements of $\mathcal{A}^n(\mathbb{R}^n)$ are functions of an $n \times n$ matrix, alternating and multilinear as a function of the columns.

Definition 1. *The $n \times n$ determinant is the element $det \in \mathcal{A}^n(\mathbb{R}^n)$, defined on an $n \times n$ matrix $A = (a_{i,j})$ by the formula*

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) a_{1,\sigma(1)} ... a_{n,\sigma(n)} \tag{1.1}$$

It is an exercise to see that indeed det belongs to $\mathcal{A}^n(\mathbb{R}^n)$, and that

$$det(e_1, ..., e_n) = 1 \ . \tag{1.2}$$

Theorem 2. *The determinant function is a basis for $\mathcal{A}^n(\mathbb{R}^n)$.*

Proof. Let $T \in \mathcal{A}^n(\mathbb{R}^n)$. Let $\phi_1, ..., \phi_n$ denote the basis for $(\mathbb{R}^n)^*$, dual to the standard basis $e_1, ..., e_n$ for $\mathbb{R}^n$. Let $\{\phi_I \mid I \in \{1, ..., n\}^n\}$, denote the corresponding basis for $L^n(\mathbb{R}^n)$. Then we may write T uniquely in the form

$$T = \sum_I c_I \phi_I \ .$$

From the alternating property it follows that $c_I = 0$ if there is an integer that appears twice in I. Thus, the only terms appearing in the expansion of T are those such that $I = \sigma$ for some $\sigma \in S_n$. Furthermore,

$$c_\sigma = sgn(\sigma) c_{(1,2,...,n)} \ .$$

Thus

$$T = c \sum_{\sigma \in S_n} sgn(\sigma) \phi_\sigma = c \ det \ ,$$

where $c = c_{(1,2,...,n)}$. This shows that det spans $\mathcal{A}^n(\mathbb{R}^n)$. Equation (1.2) shows that det is nonzero, so it is a basis. □

Theorem 3. *Let A and B be $n \times n$ matrices. Then*

1. $det(AB) = det(A)det(B)$.
2. $det(A^T) = det(A)$, *where A^T is the transpose of A.*

Proof. To prove 1, regard A as fixed, and let $T(B) = det(AB)$. Then $T \in \mathcal{A}^n(\mathbb{R}^n)$, so there exists a constant c, such that for all B,

$$T(B) = c\ det(B)\ .$$

From the definition of T, $T(I) = det(A)$, where I is the identity matrix. This shows that $c = det(A)$, which is what we wanted to prove.

To prove 2, observe that as σ varies over S_n, so does σ^{-1}. Furthermore, $sgn(\sigma) = sgn(\sigma^{-1})$. Then we have

$$\begin{aligned} det(A) &= \sum_{\sigma \in S_n} sgn(\sigma) a_{1,\sigma(1)} ... a_{n,\sigma(n)} \\ &= \sum_{\sigma \in S_n} sgn(\sigma^{-1}) a_{\sigma^{-1}(1),1} ... a_{\sigma^{-1}(n),n} \\ &= \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma(1),1} ... a_{\sigma(n),n} = det(A^T)\ . \end{aligned} \tag{1.3}$$

□

2. Basis for $\mathcal{A}^k(\mathbb{R}^n)$

Let $\mu = (i_1, ..., i_k) \in \{1, ..., n\}^n$ be a strictly increasing multiiindex. Then we get an element $\psi_\mu \in \mathcal{A}^k(\mathbb{R}^n)$ by defining, for all sequences $(v_1, ..., v_k) \in (\mathbb{R}^n)^k$,

$$\psi_\mu(v_1, ..., v_k) = det((\phi_{i_l}(v_j))_{l=1,...,k,j=1,...,k})\ . \tag{2.1}$$

Another way of writing this is

$$\psi_\mu = \sum_{\sigma \in S_n} sgn(\sigma) \phi_{\mu^\sigma}\ . \tag{2.2}$$

Theorem 4. *Let $\mathcal{I} \subset \{1, ..., n\}^k$ denote the set of all strictly increasing multiindices. Then $\{\psi_\mu \mid \mu \in \mathcal{I}\}$ is a basis for $\mathcal{A}^k(\mathbb{R}^n)$.*

Proof. Let $T \in \mathcal{A}^k(\mathbb{R}^n)$. As before, write T in the form

$$T = \sum_I c_I \phi_I\ ,$$

with the sum now running over multiindices of length k. From the alternating property it follows that $c_I = 0$ if there is an integer that appears twice in I. Thus, the only terms appearing in the expansion of T are those such that $I = \mu^\sigma$ for some $\mu \in \mathcal{I}$ and $\sigma \in S_k$. Furthermore,

$$c_{\mu^\sigma} = sgn(\sigma) c_\mu \ .$$

Thus

$$T = \sum_{\mu \in \mathcal{I}} c_\mu \sum_{\sigma \in S_n} sgn(\sigma) \phi_{\mu^\sigma} = \sum_{\mu \in \mathcal{I}} c_\mu \psi_\mu \ .$$

This shows that $\{\psi_\mu\}$ spans $\mathcal{A}^k(\mathbb{R}^n)$. Furthermore, if $\mu = (i_1, ..., i_k)$, then $c_\mu = T(e_{i_1}, ..., e_{i_k})$. This shows that $\{\psi_\mu\}$ are linearly independent. □

3. The wedge product

Theorem 5. *Let V be a finite-dimensional vector space. Then there exists a unique collection of bilinear maps:*

$$\wedge : \mathcal{A}^k(V) \times \mathcal{A}^l(V) \to \mathcal{A}^{k+l}(V) \ , \tag{3.1}$$

($\wedge(\phi, \psi)$ is written $\phi \wedge \psi$,) such that

1. *(associativity) $(f \wedge g) \wedge h = f \wedge (g \wedge h)$,*
2. *Given $T_1, ..., T_k \in \mathcal{A}^1(V)$, and given $v_1, ..., v_k \in V$,*

$$(T_1 \wedge T_2 \wedge \cdots \wedge T_k)(v_1, v_2, ..., v_k) = det(T_i(v_j)) \ . \tag{3.2}$$

Moreover, for $\phi \in \mathcal{A}^k(V)$ and $\psi \in \mathcal{A}^l(V)$,

$$\phi \wedge \psi = (-1)^{kl} \psi \wedge \phi \tag{3.3}$$

Proof. We leave it is an exercise to check that if the theorem holds for a given vector space V, then it holds for every vector space W isomorphic to V. Thus we may assume that $V = \mathbb{R}^n$. To define $\wedge$, it is sufficient to define $\psi_\mu \wedge \psi_\nu$, where μ and ν are increasing multiindices. Define the *concatenation* of two multiindices, $\mu * \nu$, to be simply the juxtaposition of μ and ν with the erasure of the obvious internal parentheses. If the entries of $\mu * \nu$ are distinct integers, define $[\mu * \nu]$ to be the multiiindex obtained by rearranging $\mu * \nu$ in increasing order. There is unique permutation σ such that

$$(\mu * \nu)^\sigma = [\mu * \nu] \ . \tag{3.4}$$

In that case, define

$$\psi_\mu \wedge \psi_\nu = sgn(\sigma) \psi_{[\mu * \nu]} \ . \tag{3.5}$$

If there is an integer that appears twice in $\mu * \nu$ define $\psi_\mu \wedge \psi_\nu = 0$. To prove associativity, it suffices by bilinearity to establish that for all increasing multiindices μ, ν, λ,

$$\psi_\mu \wedge (\psi_\nu \wedge \psi_\lambda) = (\psi_\mu \wedge \psi_\nu) \wedge \psi_\lambda \ .$$

It is an exercise to check that both sides are zero unless the entries of $\mu * \nu * \lambda$ are distinct. Assume then that the entries are distinct. There are permutations σ, τ such that

$$(\mu * \nu * \lambda)^\sigma = [\mu * \nu] * \lambda \tag{3.6}$$

$$([\mu * \nu] * \lambda)^\tau = [\mu * \nu * \lambda] \ . \tag{3.7}$$

Then

$$(\psi_\mu \wedge \psi_\nu) \wedge \psi_\lambda = sgn(\tau) sgn(\sigma) \psi_{[\mu * \nu * \lambda]} \ .$$

Similarly, there are permutations ζ, ξ, such that

$$(\mu * \nu * \lambda)^\zeta = \mu * [\nu * \lambda] \tag{3.8}$$

$$(\mu * [\nu * \lambda])^\xi = [\mu * \nu * \lambda] \ . \tag{3.9}$$

Then

$$\psi_\mu \wedge (\psi_\nu \wedge \psi_\lambda) = sgn(\xi) sgn(\zeta) \psi_{[\mu * \nu * \lambda]} \ .$$

But $\sigma \circ \tau = \zeta \circ \xi$, for they are both equal to the unique permutation that transforms $\mu * \nu * \lambda$ to $[\mu * \nu * \lambda]$. This establishes associativity, since $sgn(\sigma \circ \tau) = sgn(\sigma) sgn(\tau)$.

To see that the operation we have defined satisfies equation (3.3), observe that if μ and ν are increasing multiindices of lengths k and l respectively, such that $\mu * \nu$ has no repeats, then the permutation that transforms $\mu * \nu$ to $\nu * \mu$ has sign $(-1)^{kl}$.

With formula (3.3) in hand, we can prove (3.2) by duality. That is, fix $v_1, ..., v_k$, and think of both sides of (3.2) as functions of $T_1, ..., T_k$. As such, both sides of the equation are alternating and multilinear. (Here we use the fact that the determinant is an alternating multilinear function of the *rows*.) Thus it suffices to establish formula (3.2) when $T_i = \phi_{j_i}$ and $\mu = (j_1, ..., j_k)$ is increasing. Then $\phi_{j_1} \wedge \cdots \wedge \phi_{j_k} = \psi_\mu$, in which case formula (3.2) reproduces the definition of ψ_μ.

Formula (3.2) also establishes the uniqueness of the wedge product, since it gives an explicit formula for it.

□

Proposition 6. *Let $S : V \to W$ be a linear transformation between two finite-dimensional vector spaces. Then for all $\phi \in \mathcal{A}^k(W)$ and*

$\psi \in \mathcal{A}^l(W)$,

$$S^*(\phi \wedge \psi) = S^*(\phi) \wedge S^*(\psi) \ . \tag{3.10}$$

Proof. Let $T_1, ..., T_n$ be a sequence of elements of W^*. Then for all $v_1, ..., v_n \in V$,

$$\begin{aligned} &S^*(T_1 \wedge \cdots \wedge T_n)(v_1, ..., v_n) \\ &= (T_1 \wedge \cdots \wedge T_n)(S(v_1), ..., S(v_n)) \\ &= det(T_i(S(v_j))) \\ &= det(S^*(T_i)(v_j)) \\ &= S^*(T_1) \wedge \cdots \wedge S^*(T_n)(v_1, ..., v_n) \end{aligned} \tag{3.11}$$

Then, if $\phi = T_1 \wedge \cdots \wedge T_k$ and $\psi = T_{k+1} \wedge \cdots \wedge T_n$, we have

$$\begin{aligned} S^*(\phi \wedge \psi) &= (S^*(T_1) \wedge \cdots \wedge S^*(T_k)) \wedge (S^*(T_{k+1}) \wedge \cdots \wedge S^*(T_n)) \\ &= S^*(T_1 \wedge \cdots \wedge T_k) \wedge S^*(T_{k+1} \wedge \cdots \wedge T_n) \\ &= S^*(\phi) \wedge S^*(\psi) \end{aligned} \tag{3.12}$$

The general case follows, since both sides of equation (3.10) are bilinear as functions of ϕ and ψ. □