

THE INVERSE FUNCTION THEOREM

The following lemma is useful in several contexts.

Lemma 1. *Let k be a nonnegative integer. Let $\mathcal{U} \subset \mathbb{R}^m$ be an open set, and let $\phi : \mathcal{U} \rightarrow \mathbb{R}^n$ be a function.*

1. *If there exists a C^k , matrix-valued function $G : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}^{n,m}$, such that for all $x \in \mathcal{U}$ and $y \in \mathcal{U}$,*

$$\phi(y) - \phi(x) = G(x, y)(y - x) , \quad (0.1)$$

then ϕ is C^{k+1} .

2. *If ϕ is C^{k+1} and \mathcal{U} is convex, then a C^k , matrix-valued function G satisfying equation (0.1) exists. Moreover, for all $x \in \mathcal{U}$, $D\phi(x) = G(x, x)$.*

Proof. Statement 1 follows immediately from the definition of derivative:

$$\lim_{y \rightarrow x} \frac{\phi(y) - \phi(x) - G(x, x)(y - x)}{|y - x|} = \lim_{y \rightarrow x} \frac{(G(x, y) - G(x, x))(y - x)}{|y - x|} = 0 .$$

Therefore $D\phi(x)$ exists and is equal to the C^k function $G(x, x)$.

To prove 2, take x and y in \mathcal{U} , and let $g(t) = \phi(x + t(y - x))$. Then g is C^1 , so we may apply the fundamental theorem of calculus.

$$\phi(y) - \phi(x) = g(1) - g(0) = \int_0^1 g'(t) dt = \int_0^1 D\phi(x + t(y - x))(y - x) dt .$$

Thus,

$$\phi(y) - \phi(x) = G(x, y)(y - x) ,$$

where

$$G(x, y) = \int_0^1 D\phi(x + t(y - x)) dt . \quad (0.2)$$

Since $D\phi(x + t(y - x))$ is a C^k function of x , y and t , we see that $G(x, y)$ is C^0 . If $k > 1$, we may differentiate under the integral sign, so $G(x, y)$ is C^k . \square

Theorem 2 (Inverse Function Theorem). *Let $\mathcal{U} \subset \mathbb{R}^n$ be an open neighborhood of a point a , and let $\phi : \mathcal{U} \rightarrow \mathbb{R}^n$ be a C^1 function such that $D\phi(a)$ is invertible. Then there are open neighborhoods \mathcal{V} and \mathcal{W}*

of a and $\phi(a)$ respectively, such that $\phi|_{\mathcal{V}}$ is an invertible function from \mathcal{V} onto \mathcal{W} , the inverse of which is also \mathcal{C}^1 .

Proof. By lemma 1 there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of a , and a continuous, matrix-valued function $G : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}^{n,m}$, such that for all $x \in \mathcal{V}$ and $y \in \mathcal{V}$,

$$\phi(y) - \phi(x) = G(x, y)(y - x) . \quad (0.3)$$

Then $G(a, a) = D\phi(a)$ is invertible. By shrinking \mathcal{V} if necessary, we may then assume that $G(x, y)$ is invertible for all x and y in \mathcal{V} . Then it follows immediately from equation (0.3) that ϕ is one-to-one.

Let $\mathcal{W} = \phi(\mathcal{V})$. Claim: \mathcal{W} is open. To prove the claim, let $x \in \mathcal{V}$. We must find an open ball around $\phi(x)$, contained in \mathcal{W} . By composing ϕ with affine isomorphisms on both sides, we can assume $x = \phi(x) = 0$ and $D\phi(0) = I$. Then $G(0, 0) = I$. By the continuity of G , we may choose a number $r > 0$, small enough that $|I - G(z, y)| < 1/2$ for all z and y in the closed ball $B_{2r}(0)$. Let $y \in B_r(0)$ be an arbitrary point. Let

$$f(z) = z - \phi(z) + y .$$

Then for all z and w in $B_{2r}(0)$,

$$|f(z) - f(w)| = |z - w - G(w, z)(z - w)| < \frac{1}{2}|z - w| .$$

In particular, if $|z - y| \leq r$, then $|f(z) - f(0)| = |f(z) - y| < \frac{1}{2}|z| < \frac{1}{2}2r = r$. Then f is a contraction mapping from $\bar{B}_r(y)$ to itself, and therefore has a fixed point, z . Then $\phi(z) = y$. This proves that \mathcal{W} is open.

Let $\psi = \phi^{-1} : \mathcal{W} \rightarrow \mathcal{V}$. The preceding paragraph shows that ψ^{-1} takes open sets to open sets. That is, ψ is continuous. Now rewrite equation (0.3) in the form

$$w - z = G(\psi(z), \psi(w))(\psi(w) - \psi(z)) , \quad (0.4)$$

for all $w, z \in \mathcal{W}$. Then

$$\psi(w) - \psi(z) = H(z, w)(w - z) ,$$

where H is the continuous function

$$H(z, w) = G(\psi(z), \psi(w))^{-1} .$$

This shows that ψ is \mathcal{C}^1 .

□

Remark: By a bootstrapping argument, one can show that if ϕ in theorem 2 is C^k , for $k > 1$, then ϕ^{-1} is also C^k . The proof is left to the reader.