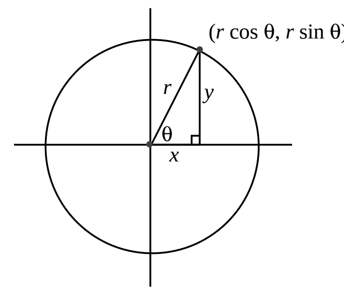
Trigonometry, New and Old
MATH 2400(H)
Fall 2007

Definitions:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$



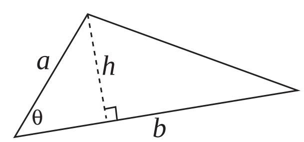
Note that cos and sec are **even** functions, whereas sin, tan, csc, and cot are **odd** functions. The basic relations are, of course, based on the Pythagorean Theorem:

$$\begin{array}{ccccc} \sin^2\theta & + & \cos^2\theta & = & 1 \\ \tan^2\theta & + & 1 & = & \sec^2\theta \\ 1 & + & \cot^2\theta & = & \csc^2\theta \end{array}$$

Note also that $\sin(\pi/2-\alpha) = \cos\alpha$, $\cos(\pi/2-\alpha) = \sin\alpha$, $\sin(\pi-\alpha) = \sin\alpha$, and $\cos(\pi-\alpha) = -\cos\alpha$.

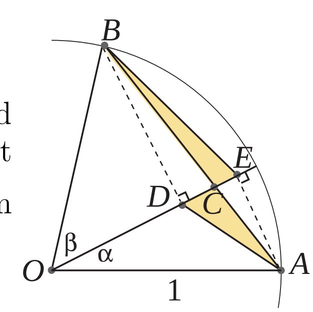
Area: The area of a triangle is $A = \frac{1}{2}bh$. If two sides, a and b, and the included angle, θ, of a triangle are given, then we take b as the base and the height is $h = a\sin\theta$, so

$$\boxed{A = \tfrac{1}{2}ab\sin\theta\,.}$$



Addition Formulas: We begin with

(†) $$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\,.}$$



Consider the diagram at the right. Here points D and E, resp., are obtained by dropping perpendiculars from B and A, resp., onto $\overline{OC}$. It follows that $\triangle ACE \sim \triangle BCD$, and so $\dfrac{AE}{BD} = \dfrac{CE}{CD}$. Thus, $(AE)(CD) = (BD)(CE)$, from which we conclude that $\text{area}\triangle ACD = \text{area}\triangle BCE$.

Now, using this result, we have[1]

$$\begin{aligned} \sin(\alpha+\beta) &= 2\,\text{area}\triangle OAB = 2\big(\text{area}\triangle OAC + \text{area}\triangle OCB\big) \\ &= 2\big(\text{area}\triangle OAD + \text{area}\triangle OEB\big) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\,. \end{aligned}$$

[1]This proof was inspired by a suggestion of Dr. Mark Faucette at the University of West Georgia.

By substituting $-\beta$ for β in (†) and using the symmetries of the respective functions, we obtain

$$\boxed{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta\,.}$$

Similarly, substituting $\frac{\pi}{2} \pm \alpha$ for α in (†), we obtain

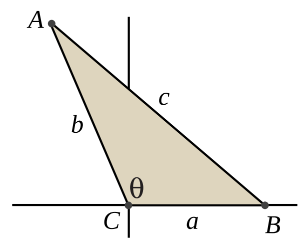
$$\boxed{\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta\,.}$$

As a consequence we have the <u>double angle formulas</u>:

$$\boxed{\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha}$$
$$\boxed{\sin 2\alpha = 2\sin\alpha\cos\alpha}$$

<u>Law of Cosines</u>: In $\triangle ABC$, let c be the length of the side opposite angle θ. Then

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta\,.}$$

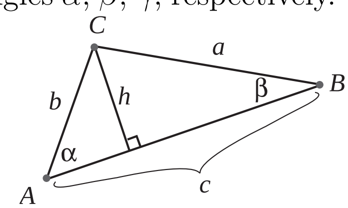


The easiest proof is to assign cartesian coordinates to the points A, B, and C. Let $C = (0,0)$, $B = (a,0)$, and $A = (b\cos\theta, b\sin\theta)$. Then we infer from the distance formula that

$$c^2 = (a - b\cos\theta)^2 + (b\sin\theta)^2 = a^2 + b^2 - 2ab\cos\theta\ .$$

<u>Law of Sines</u>: In $\triangle ABC$, let the sides a, b, c be opposite the angles α, β, γ, respectively. Then

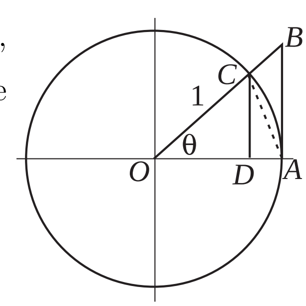
$$\boxed{\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}\,.}$$



We'll establish only the first equality here. Drop a perpendicular from C to $\overline{AB}$, as indicated in the diagram. Then

$$h = b\sin\alpha = a\sin\beta \implies \frac{\sin\alpha}{a} = \frac{\sin\beta}{b}\ ,$$

as desired.



<u>Basic Limits</u>: Recall that the area of the unit circle is (by definition) π, and so the area of a sector with central angle θ is $\dfrac{\theta}{2\pi}\pi = \dfrac{1}{2}\theta$. From the figure at the right, we see first that $\quad \text{area}\triangle OAC \le \text{area}(\text{sector}OAC), \quad$ and so

$$|\sin\theta| \le |\theta|\,.$$

From this we conclude that

$$1 - \cos\theta = (1 - \cos\theta)\cdot\frac{1+\cos\theta}{1+\cos\theta} = \frac{\sin^2\theta}{1+\cos\theta} \le \theta^2 \quad \text{for } |\theta| \le \frac{\pi}{2}\,.$$

Thus,

$$\boxed{\lim_{\theta\to 0}\sin\theta = 0 \qquad \text{and} \qquad \lim_{\theta\to 0}\cos\theta = 1\,.}$$

Exercise: Prove that sin and cos are continuous. Deduce that all the trigonometric functions are continuous.

Next, we have

$$\text{area}\triangle ODC \le \text{area}(\text{sector}OAC) \le \text{area}\triangle OAB$$

and so

$$\frac{1}{2}(\sin\theta)(\cos\theta) \le \frac{1}{2}\theta \le \frac{1}{2}\tan\theta\,,$$

from which we conclude that

$$\cos\theta \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}\,.$$

By the squeeze principle, since $\lim_{\theta\to 0}\cos\theta = 1$, we have

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\,.}$$

Exercise: Prove that $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$.

Differentiation Formulas:

$\sin'(x)$	$=$	$\cos x$	$\cos'(x)$	$=$	$-\sin x$
$\tan'(x)$	$=$	$\sec^2 x$	$\cot'(x)$	$=$	$-\csc^2 x$
$\sec'(x)$	$=$	$\sec x\tan x$	$\csc'(x)$	$=$	$-\csc x\cot x$

We prove the first and leave the rest to the reader.

$$\begin{aligned}\sin'(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h} = \lim_{h\to 0}\left(\sin x\cdot\frac{\cos h - 1}{h} + \cos x\cdot\frac{\sin h}{h}\right)\\ &= \sin x\lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h} = \sin x\cdot 0 + \cos x\cdot 1 = \cos x\,,\end{aligned}$$

by the limits above.

A remark on terminology: If f is a trig function, define the co-function cof by $cof(x) = f(\pi/2 - x)$ (i.e., the function of the complementary angle). If $f' = g$, then the chain rule gives $(cof)'(x) = -f'(\pi/2 - x) = -cog$. This means we need only commit to memory the first column of the rules above.