MATH 2200 Exam 2 Solutions

Instructor: Dr. Shuzhou Wang

Print Your Name:

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UGA Academic Honesty Policy applies. **No Calculators.** Closed Book. *Show work, otherwise no credit will be given.* You loose points for giving confusing arguments. Cross out the parts you do not want to be graded. The last problem is optional. The rest problems are worth 100 points (i.e. 100%).

Problem #	Points	Score
1	28	
2	22	
3	25	
4	25	
bonus	15	
Total	115	

1. (28 points) Find the derivative of each of the following functions (you can use any method you want and you do not need to simplify your final answers):

(a) $y = (x^2 - 1)^3(x^3 - 4x^2 + 5x - 6)$

$$y' = [(x^2 - 1)^3]'(x^3 - 4x^2 + 5x - 6) + (x^2 - 1)^3(x^3 - 4x^2 + 5x - 6)',$$
$$y' = [3(x^2 - 1)^2 2x](x^3 - 4x^2 + 5x - 6) + (x^2 - 1)^3(3x^2 - 8x + 5).$$

(b) $y = \dfrac{3x^2 + 4}{x^2 - 4}$

$$y' = \frac{(3x^2 + 4)'(x^2 - 4) - (3x^2 + 4)(x^2 - 4)'}{(x^2 - 4)^2},$$
$$y' = \frac{6x \cdot (x^2 - 4) - (3x^2 + 4) \cdot 2x}{(x^2 - 4)^2} = \frac{-32x}{(x^2 - 4)^2}.$$

(c) $y = (3x^2 + 1)^3(5x - 1)^5$

$$y' = [(3x^2 + 1)^3]' \cdot (5x - 1)^5 + (3x^2 + 1)^3 \cdot [(5x - 1)^5]',$$
$$y' = [3(3x^2 + 1)^2 \cdot 6x](5x - 1)^5 + (3x^2 + 1)^3 \cdot [5(5x - 1)^4 \cdot 5]$$

(d)

$$y = [(10x+1)^5 + 2x]^{100}$$
$$y' = 100[(10x+1)^5 + 2x]^{99}[(10x+1)^5 + 2x]',$$
$$y' = 100[(10x+1)^5 + 2x]^{99}[[(10x+1)^5]' + 2],$$
$$y' = 100[(10x+1)^5 + 2x]^{99}[5(10x+1)^4 \cdot 10 + 2].$$

2. (22 points) Air is being pumped into a spherical balloon in such a way that its radius is increasing at a constant rate of $\frac{1}{5}$ cm/s. What is the time rate of change, in cubic centimeters per second, of the volume of the balloon when the radius is 10 cm? (Express the answer in an exact number–Points will be taken off for an approximate number. For instance, 3.14 is an approximation of π so do not used it here. The volume of a sphere is $V = \frac{4}{3}\pi r^3$)

$$V = \frac{4}{3}\pi r^3,$$
$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

When $r = 10$,

$$\frac{dV}{dt} = 4\pi \cdot 10^2 \cdot (1/5) = 80\pi (cm^3/s).$$

3. (25 points) Find the maximum and minimum values attained by $f(x) = 2x^3 - 3x^2 - 12x + 15$ on the closed interval $[0, 3]$. **Hint:** Be cautious about the domain $[0, 3]$.

$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2),$$
$$f'(x) = 6(x-2)(x+1),$$
$$x = 2. \quad (x = -1 \quad \text{discarded})$$

$$\begin{aligned} f(2) &= 2 \cdot 2^3 - 3 \cdot 2^2 - 12 \cdot 2 + 15 \\ &= 2^2\,[2^2 - 3] - 2x + 15 \\ &= -5 \leftarrow \text{min} \end{aligned}$$

$$\begin{aligned} f(0) &= 15 \leftarrow \text{max.} \\ f(3) &= 2 \cdot 3^3 - 3 \cdot 3^2 - 12 \cdot 3 + 15 \\ &= 3^3\,[2 - 1] - 36 + 15 \\ &= 6 \end{aligned}$$

4. (25 points) A manufacturer needs to design a rectangular box that has a square base with edges at least 2 inches long. The box has no top, and the total area of its five sides is 300 square inches. What are the dimensions of the box with maximal volume?

Let x be the length of a base of the box, h the height, V the volume. Then

$V = x^2 h,$

$x^2 + 4xh = 300, \qquad \text{so} \qquad 2 \le x \le \sqrt{300},$

$h = \dfrac{300 - x^2}{4x}.$

$V = x \cdot \dfrac{300 - x^2}{4x} = \dfrac{1}{4}x(300 - x^2), \qquad \text{or} \qquad V = \dfrac{1}{4}(300x - x^3), \qquad 2 \le x \le \sqrt{300}.$

$\dfrac{dV}{dx} = \dfrac{1}{4}(300 - 3x^2) = 0, \quad x = 10, \quad (x = -10 \text{ discarded}),$

$h = \dfrac{300 - 10^2}{4 \times 10} = 5,$

$V(10) = \dfrac{1}{4} \cdot 10 \cdot (300 - 10^2) = 500, \qquad \leftarrow \text{max volume},$

$V(2) = \dfrac{1}{4} \cdot 2 \cdot [300 - 2^2] = \dfrac{592}{4},$

$V(\sqrt{300}) = \dfrac{1}{4} \cdot \sqrt{300} \cdot [300 \cdot -(\sqrt{300})^2] = 0.$

Dimensions of the box with max volume (500 in.^{3}) are

$x = 10$ in., $h = 5$ in.

5. * (15 bonus points) (Note: This **Optional** problem will be graded strictly.)
An arbitrary cylinder is inscribed in a cone of radius 3 feet and height 6 feet.

(a) Express the volume of the cylinder as a function of the radius of the cylinder alone.

(b) Find the maximal possible volume of the cylinder.

(a) Let r and h be respectively the radius and height of an inscribed cylinder, and V its volume. Then $V = \pi r^2 h$.

Draw a cross section from the apex of the cone that passes the center of its base. We have similar triangles (see separate page for graph).

$$\frac{h}{3-r} = \frac{6}{3}, \quad h = 2(3-r),$$

Plug this into the above formula for V, we get

$$V = \pi r^2 \cdot 2(3-r) = 2\pi(3r^2 - r^3), \quad 0 \le r \le 3.$$

(b) $V'(r) = 2\pi(6r - 3r^2)$. Set $2\pi(6r^{-}3r^2) = 0$. Then, $r = 0, \quad r = 2$.

$V(2) = 8\pi \quad \leftarrow$ maximal volume.

$V(0) = 0,$

$V(3) = 0.$

Graph for Math 2200 Exam 2. Problem 5.