MATH 8150 Final Exam — Spring 2003

Instructor: Dr. Shuzhou Wang

Print Your Name: ______________________________

Arrange your solutions orderly.
Label each problem clearly.
Cross out the parts you do not want to be graded.

You can freely use the theorems we have covered in the semester (either proved in the text or in class).
You can use results in homework if you state them clearly.

Hints and Solutions will be posted on the course web page
http://www.math.uga.edu/~szwang/teaching/8150-s03.html.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

!!! Good Luck and Have Fun!!!

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1. Let $f(z) = u + iv$ be an entire function such that $u(x, y) = Re(f(x + iy))$ is a polynomial in x, y. Show that $f(z)$ is a polynomial in z.

2. Let $a_n(z)$ be a sequence of analytic functions on the unit disk $D : |z| < 1$ such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed subsets of D. Show that $\sum_{n=0}^{\infty} |a'_n(z)|$ converges uniformly on bounded and closed subsets of D.

   Hint: It suffices to show that for fixed $0 < r < 1$, $\sum_{n=0}^{\infty} |a'_n(z)|$ converges uniformly on $|z| \leq r$. Use Cauchy's derivative formula appropriately.

3. Let $f(z)$ be a non-constant entire function. Let $w_0 = f(z_0)$. Let $R > 0$ be any positive number. Show that $f(z) - w_0 = 0$ has only a finite number of solutions in the disk of radius R centered at the origin.

4. Evaluate $\int_0^{\infty} \frac{x \sin x}{1 + x^2} dx$. Justify all steps (including convergence of the integral).

5. Let f be analytic on a bounded domain D that is continuous and nowhere zero on the closure $\overline{D}$. Show that if $|f(z)| = M$ (a constant) for $z \in \partial D$, the boundary of D, then $f(z) = e^{i\theta} M$ (where θ is a real constant).

6. **The Local Mapping Theorem:** Let $f(z)$ be analytic in a domain D. Fix $z_0 \in D$ and let $w_0 = f(z_0)$. Suppose z_0 is a zero of finite order m (i.e. multiplicity m) for $f(z) - w_0 = 0$. Show that there exist $\delta > 0$ and $\tilde{\delta} > 0$ such that for each w with $0 < |w - w_0| < \tilde{\delta}$, the equation $f(z) - w = 0$ has exactly m *distinct* solutions inside the disk $|z - z_0| < \delta$.

7. (1) Let $P(z), Q(z)$ be polynomials in z of degrees m and n respectively with $m < n$. Assume that the roots z_k of $Q(z)$ are distinct ($k = 1, \cdots, n$) and they are not roots of $P(z)$. Show that

$$P(z)/Q(z) = \sum_{k=1}^{n} c_k/(z - z_k)$$

   for some c_k and express c_k in terms of P and Q.

   (2) Assume that only one of the roots z_k has order 2. Formulate a similar conclusion as in part (1) above and prove it. Hint: Part (2) is related to a homework problem on the principal part of $P(z)/Q(z)$.

8. Let $G = \{z : \ |z - i| < \sqrt{2}, \ |z + i| < \sqrt{2}\} \backslash [0, 1)$. Find a bijective conformal map from G to the upper half plane.