ALGEBRA QUALIFYING EXAM, FALL 2011

1. Let G be a group with order $595 = 5 \cdot 7 \cdot 17$, and let H be a 5-sylow subgroup of G. Prove that H is normal in G, and that H is contained in the center of G.

2. (a) Let $\mathbb{F}$ be a field. Define the group $\mathrm{SL}(n, \mathbb{F})$.
   (b) Let $\mathbb{F}_q$ be a finite field with q elements. Find the order of $\mathrm{SL}(n, \mathbb{F}_q)$.
   (c) Find the 2-Sylow subgroup of $\mathrm{SL}(2, \mathbb{F}_3)$.

3. Let K be a field of characteristic 0, and let A be an $n \times n$ matrix with coefficients in K. Show that A is nilpotent if and only if $Tr(A^k) = 0$ for $k = 1, \ldots, n$.

4. (a) Let $T \colon V \to W$ be a linear map of linear spaces over a field k. Define the induced dual map $T^* \colon W^* \to V^*$ between the dual spaces W^*, V^*, without using bases.
   (b) Let $\{e_1, \ldots, e_n\}$ be a basis of V, $\{f_1, \ldots, f_m\}$ a basis of W, and A be the matrix of T in these bases. Find the matrix of T^* in the dual bases $\{f_j^*\}$, $\{e_i^*\}$.
   (c) Now, let k be algebraically closed, V be a finite-dimensional vector space, and $T \colon V \to V$. Prove that T and T^* have isomorphic Jordan normal forms.

5. Prove the following generalization of the classical Chinese Remainder Theorem: Let R be a commutative ring with identity and let I, J be two ideals. Suppose that $I + J = R$. Then
   (a) $R/(I \cap J) = R/I \oplus R/J$.
   (b) $I \cap J = IJ$, where IJ denotes the ideal generated by all products ij with $i \in I$, $j \in J$.

6. Let R be a not necessarily commutative ring which perhaps does not have a unit. Suppose that there is some element $r \in R$ such that for every element $y \in R$ there exists $x \in R$ with $xr = y$ (i.e. every element y is divisible by r on the right).
   Show that there exists a left ideal I of R which is maximal (i.e. maximal with respect to the property $I \neq R$).

7. (a) Suppose G is a group and N_1, N_2 are two subgroups of index 2 with $N_1 \neq N_2$. Show that $N_1 \cap N_2$ is a normal subgroup of G of index 4.
   (b) Suppose we have fields F, K_1, K_2, L with inclusions:

$$\begin{tikzcd}
 & L & \\
K_1 \arrow[ur, hook] & & K_2 \arrow[ul, hook'] \\
 & F \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}$$

   Supppose that L/F, K_1/F and K_2/F are Galois extensions with $[K_1 : F] = 2 = [K_2 : F]$. If E is the smallest subfield of L which contains both K_1 and K_2, show that E/F is Galois.

8. Let K be the splitting field of $x^4 - x^2 - 3$ over $\mathbb{Q}$. Find $\mathrm{Gal}(K/\mathbb{Q})$, its subgroups, and describe the Galois correspondence between the subgroups of $\mathrm{Gal}(K/\mathbb{Q})$ and the subfields of K.