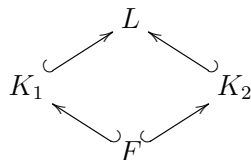


ALGEBRA QUALIFYING EXAM, FALL 2011

1. Let G be a group with order $595 = 5 \cdot 7 \cdot 17$, and let H be a 5-sylow subgroup of G . Prove that H is normal in G , and that H is contained in the center of G .
2. (a) Let \mathbb{F} be a field. Define the group $\text{SL}(n, \mathbb{F})$.
 (b) Let \mathbb{F}_q be a finite field with q elements. Find the order of $\text{SL}(n, \mathbb{F}_q)$.
 (c) Find the 2-Sylow subgroup of $\text{SL}(2, \mathbb{F}_3)$.
3. Let K be a field of characteristic 0, and let A be an $n \times n$ matrix with coefficients in K . Show that A is nilpotent if and only if $\text{Tr}(A^k) = 0$ for $k = 1, \dots, n$.
4. (a) Let $T: V \rightarrow W$ be a linear map of linear spaces over a field k . Define the induced dual map $T^*: W^* \rightarrow V^*$ between the dual spaces W^*, V^* , without using bases.
 (b) Let $\{e_1, \dots, e_n\}$ be a basis of V , $\{f_1, \dots, f_m\}$ a basis of W , and A be the matrix of T in these bases. Find the matrix of T^* in the dual bases $\{f_j^*\}, \{e_i^*\}$.
 (c) Now, let k be algebraically closed, V be a finite-dimensional vector space, and $T: V \rightarrow V$. Prove that T and T^* have isomorphic Jordan normal forms.
5. Prove the following generalization of the classical Chinese Remainder Theorem: Let R be a commutative ring with identity and let I, J be two ideals. Suppose that $I + J = R$. Then
 (a) $R/(I \cap J) = R/I \oplus R/J$.
 (b) $I \cap J = IJ$, where IJ denotes the ideal generated by all products ij with $i \in I, j \in J$.
6. Let R be a not necessarily commutative ring which perhaps does not have a unit. Suppose that there is some element $r \in R$ such that for every element $y \in R$ there exists $x \in R$ with $xr = y$ (i.e. every element y is divisible by r on the right).
 Show that there exists a left ideal I of R which is maximal (i.e. maximal with respect to the property $I \neq R$).
7. (a) Suppose G is a group and N_1, N_2 are two subgroups of index 2 with $N_1 \neq N_2$. Show that $N_1 \cap N_2$ is a normal subgroup of G of index 4.
 (b) Suppose we have fields F, K_1, K_2, L with inclusions:



Suppose that $L/F, K_1/F$ and K_2/F are Galois extensions with $[K_1 : F] = 2 = [K_2 : F]$. If E is the smallest subfield of L which contains both K_1 and K_2 , show that E/F is Galois.

8. Let K be the splitting field of $x^4 - x^2 - 3$ over \mathbb{Q} . Find $\text{Gal}(K/\mathbb{Q})$, its subgroups, and describe the Galois correspondence between the subgroups of $\text{Gal}(K/\mathbb{Q})$ and the subfields of K .