Algebra Preliminary Examination

August 2012

1) (10 pts) Let G be a finite group and X be a G-set (i.e, G acts on X)

a) Let $x \in X$ and $G_x = \{g \in G : \ g.x = x\}$. Show that G_x is a subgroup of G.

b) Let $x \in X$ and $G \cdot x = \{g.x : \ g \in G\}$. Prove that there is a bijection between elements in $G \cdot x$ and the left cosets of G_x in G.

2) (10 pts) Let G be a group of order 30.

a) Show that G contains normal subgroups of order 3, 5, and 15.

b) Give presentations and relations for possible G (up to isomorphism).

c) Determine how many groups of order 30 there are up to isomorphism.

3) (10 pts) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 5. Assume that $f(x)$ has all but two roots in $\mathbb{R}$ (real numbers). Compute the Galois group of $f(x)$ over $\mathbb{Q}$. Justify your answer.

4) (10 pts) Let $f(x)$ be a polynomial in $\mathbb{Q}[x]$ and K be a splitting field of $f(x)$ over $\mathbb{Q}$. Assume that $[K : \mathbb{Q}] = 1225$. Show that $f(x)$ is solvable by radicals.

5) (10 pts) Let U be an infinite-dimensional vector space over a field k, $f : U \to U$ a linear map, and $u_1, \ldots, u_m \in U$ vectors such that U is generated by $u_1, \ldots, u_m$ and $f^d(u_1), \ldots, f^d(u_m)$, $d \in \mathbb{N}$.

Prove that U can be written as a direct sum $U \cong V \oplus W$ of two vector subspaces such that

(1) V has a basis consisting of some vectors $v_1, \ldots, v_n$ and $f^d(v_1), \ldots, f^d(v_n)$, $d \in \mathbb{N}$.

(2) W is finite-dimensional,

Moreover, prove that for any other such decomposition $U \cong V' \oplus W'$, one has $W' \cong W$.

6) (10 pts) Let R be a ring, and M be an R-module. Recall that M is called *Noetherian* if any strictly increasing chain of submodules $M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots$ is finite. Call a proper submodule $M' \subsetneq M$ *intersection-indecomposable* if it can not be written as the intersection of two proper submodules $M' = M_1 \cap M_2$, $M_i \subsetneq M$.

Prove that for every Noetherian module M, any proper submodule $N \subsetneq M$ can be written as a finite intersection $N = N_1 \cap \cdots \cap N_k$ of intersection-indecomposable modules.

7) (10 pts) Let k be a field of characteristic 0, let $A, B \in M_n(k)$ be two square $n \times n$ matrices (with coefficients in k) such that $AB - BA = A$. Prove that $\det A = 0$.

Moreover, when the characteristic of the field k is two, find a counterexample to the aforementioned statement.

8) (10 pts) Prove that any nondegenerate matrix $X \in M_n(\mathbb{R})$ ($n \times n$ matrices with real coefficients) can be written as $X = UT$, where U is an orthogonal matrix in $M_n(\mathbb{R})$ and T is an upper triangular matrix in $M_n(\mathbb{R})$.