UGA Algebra Qualifying Examination, Fall 2014

1. Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial, and let L be a finite Galois extension of $\mathbb{Q}$. Let $f(x) = g_1(x)g_2(x) \cdots g_r(x)$ be a factorization of f into irreducibles in $L[x]$.
   (a) Prove that each of the factors $g_i(x)$ has the same degree.
   (b) Give an example to show that if L is not Galois over $\mathbb{Q}$, the conclusion of part (a) need not hold.

2. Let G be a group of order 96.
   (a) Show that G has either one or three 2-Sylow subgroups.
   (b) Show that either G has a normal subgroup of order 32 or a normal subgroup of order 16.

3. Consider the polynomial $f(x) = x^4 - 7$ in $\mathbb{Q}[x]$, and let $E/\mathbb{Q}$ be the splitting field of f.
   (a) What is the structure of the Galois group of $E/\mathbb{Q}$?
   (b) Give an explicit description of all of the intermediate subfields $\mathbb{Q} \subset K \subset E$ in the form $K = \mathbb{Q}(\alpha), \mathbb{Q}(\alpha, \beta), \ldots$, where α, β, etc. are complex numbers. Describe the corresponding subgroups of the Galois group.

4. Let F be a field and T and $n \times n$ matrix with entries in F. Let I be the ideal consisting of all polynomials $f \in F[x]$ such that $f(T) = 0$. Show that the following statements are equivalent about a polynomial $g \in I$:
   (a) g is irreducible,
   (b) if $k \in F[x]$ is nonzero and of degree strictly less than g, $k(T)$ is an invertible matrix.

5. Let T be a 5×5 complex matrix with characteristic polynomial $\chi(x) = (x-3)^5$, and minimal polynomial $m(x) = (x-3)^2$. Determine all possible Jordan forms of T.

6. Let G be a group, and let $H, K < G$ be subgroups of finite index. Show that $[G : H \cap K] \leq [G : H][G : K]$.

7. Give a careful proof that $\mathbb{C}[x, y]$ is not a principal ideal domain.

8. Let R be a commutative ring *without* unit, such that R does not contain a proper maximal ideal, and R is not the zero ring. Prove that for all $x \in R$, the ideal xR is proper. You may assume the axiom of choice.