# Algebra Prelim

Work as many problems as possible.

1. Suppose that $A, B$, and $C$ are groups and we have homomorphisms $\beta\colon A \to B$ and $\gamma\colon A \to C$. Show that if $\beta$ is surjective and if the kernel of $\beta$ is a subgroup of the kernel of $\gamma$, then there exists a homomorphism $\mu\colon B \to C$ such that $\mu \circ \beta = \gamma$.

2. Let $p$ be a prime and let $A$ be a normal subgroup of a finite group $G$. Suppose that the order of $A$ is $p$. Prove that $A$ is in the center of $G$.

3. Let $R$ be a commutative Noetherian ring. Let $M$ be an $R$-module. For $m \in M$, the annihilator of $m$ is the set $A(m) = \{x \in R \,|\, xm = 0\}$. Show that if $m \in M$, $m \neq 0$, $A(m) \neq 0$, then there exists $r \in R$ such that $rm \neq 0$ and $A(rm)$ is a prime ideal.

4. Let $V$ be a finite dimensional vector space over a field $\mathbb{F}$. Let $T\colon V \to V$ be a linear transformation.
   (a) Show that $T$ has a minimal polynomial $f(x) \in \mathbb{F}[x]$. (A minimal polynomial of $T$ is a polynomial $f(x) \in \mathbb{F}[x]$ such that $f(T) = 0$ and whenever $g(x)$ is a polynomial in $\mathbb{F}[x]$ with $g(T) = 0$, we have that $f(x)$ divides $g(x)$).
   (b) With $f(x)$ as in (a), suppose that $f(x) = g(x) \cdot h(x)$ where $g(x)$ and $h(x)$ are relatively prime. Show directly that $V = V_1 \ominus V_2$ where $V_1$ and $V_2$ are subspaces which are invariant under $T$ and such that the minimal polynomial of $T$ on $V_1$ is $g(x)$, while the minimal polynomial of $T$ on $V_2$ is $h(x)$.

5. Suppose that $G$ is a simple group of order 660. Prove that $G$ is isomorphic to a subgroup of $A_{12}$, the alternating group on 12 letters. (Hint: look at the Sylow 11-subgroups of $G$.)

6. Let $K$ be a field and suppose that $f(t) \in K[t]$ is a polynomial of degree $n$.
   (a) Define what is meant by a splitting field for $f(t)$ over $K$.
   (b) Prove that $f(t)$ has a splitting field over $K$ which is an extension of degree at most $n!$.

7. Suppose that $R$ is a ring with unit and that

$$\begin{tikzcd}
 & & & D \arrow[d, "\gamma"] & \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0
\end{tikzcd}$$

is a diagram of $R$-modules and homomorphisms with exact row. Prove that there is an $R$-module $M$ and homomorphisms $\tau, \sigma, \theta$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\sigma"] \arrow[d, equal] & M \arrow[r, "\tau"] \arrow[d, "\theta"] & D \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & D \arrow[r] & 0
\end{tikzcd}$$

has exact rows and commutes. (Hint: Let $M = \{(b,d) \in B \oplus D \,|\, \beta(b) = \gamma(d)\}$.)

8. Let $E$ be a subfield of the complex numbers $\mathbb{C}$ and suppose that $\zeta \in \mathbb{C}$ is a primitive $n$th root of 1 for some positive integer $n$.

   (a) Is $E(\zeta)$ a normal extension of $E$? Prove or give a counterexample.

   (b) If $E = \mathbb{Q}$, the rationals, what is the degree extension of $E(\zeta)$ over $E$? Explain briefly.

9. Suppose that $U$ and $V$ are subspaces of a vector space $W$ over a field $\mathbb{F}$. Suppose that $W$ has dimension $n$ and both $U$ and $V$ have dimension $s < n$. Prove that there is a subspace $X$ of dimension $n - s$ such that $X \cap U = 0 = X \cap V$. (Hint: One approach is to build a basis for $X$ by first choosing $x_1 \in W$ such that $x_1 \notin U \cup V$ (how?) then factoring out the subspace spanned by $x_1$ and choosing a second basis element, etc.)