

**Graduate Preliminary Exam, Spring 2006**  
(9 problems counted equally, 3 hours)

- # 1. i) State what it means for a sequence  $\{a_n\}$  of real numbers to be *Cauchy*.  
ii) Prove that every convergent sequence is Cauchy.

# 2. Prove that for every positive integer  $n$ ,  $n^3 + 2n$  is divisible by 3.

# 3. Find all complex numbers  $z = x + iy$ , ( $x, y \in \mathbb{R}$ ), such that  $e^z = 2i$ .

# 4. Let  $L \subset \mathbb{R}^2$  be the line spanned by the vector  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ . Let  $P \subset \mathbb{R}^2$  be the line defined by the equation  $x + 2y = 0$ . Find the standard matrix for the linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $T(v) = 0$  for  $v \in L$  and  $T(v) = v$  for  $v \in P$ .

# 5. Find an invertible matrix  $Q$  so that  $Q^{-1} \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} Q$  is diagonal.

# 6. Let  $f$  be a Riemann integrable function on a closed interval  $[a, b]$ . For  $x \in [a, b]$ , let  $g(x) = \int_a^x f(t) dt$ . Prove that  $g$  is continuous.

# 7. Let  $\tau$  be the arc of the unit circle in the first quadrant, from  $(1, 0)$  to  $(3/5, 4/5)$ . Compute

$$\int_{\tau} x dy + y dx .$$

# 8. Let  $f(x, y) = \frac{x^2 y}{x^2 + y^2}$  for  $(x, y) \neq (0, 0)$  and let  $f(0, 0) = 0$ .

- i) Prove that  $f$  is continuous at  $(0, 0)$ .  
ii) Prove that  $f$  is not differentiable at  $(0, 0)$ .

# 9. For each positive integer  $n$ , let  $f_n(x) = \frac{x}{x+n}$  for  $x \in [0, \infty)$ . Show that the sequence of functions  $\{f_n\}$  converges pointwise on  $[0, \infty)$  to the 0-function but does not converge uniformly.