**Directions:** Justify all the calculations and state the theorems you use in your answers. Each problem is worth 10 points. In the solution of a part of a problem, you may use any earlier part of that problem, whether or not you’ve correctly solved it.

1. Let $A$ be a square matrix over the complex numbers. Suppose that $A$ is nonsingular and that $A^{2019}$ is diagonalizable over $\mathbb{C}$. Show that $A$ is also diagonalizable over $\mathbb{C}$.

2. Let $F = \mathbb{F}_p$, where $p$ is a prime number.
   (a) Show that if $\pi(x) \in F[x]$ is irreducible of degree $d$, then $\pi(x)$ divides $x^{p^d} - x$.
   (b) Show that if $\pi(x) \in F[x]$ is an irreducible polynomial that divides $x^{p^n} - x$, then deg $\pi(x)$ divides $n$.

3. How many isomorphism classes are there of groups of order 45? Describe a representative from each class.

4. For a finite group $G$, let $c(G)$ denote the number of conjugacy classes of $G$.
   (a) Prove that if two elements of $G$ are chosen uniformly at random, then the probability they commute is precisely $c(G)/|G|$.
   (b) State the class equation for a finite group.
   (c) Using the class equation (or otherwise) show that the probability in part (a) is at most $\frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2} |G:Z(G)|$. Here, as usual, $Z(G)$ denotes the center of $G$.

5. Let $R$ be an integral domain. Recall that if $M$ is an $R$-module, the rank of $M$ is defined to be the maximum number of $R$-linearly independent elements of $M$.
   (a) Prove that for any $R$-module $M$, the rank of $\text{Tor}(M)$ is 0.
   (b) Prove that the rank of $M = \text{rank of } M/\text{Tor}(M)$.
   (c) Suppose that $M$ is a non-principal ideal of $R$. Prove that $M$ is torsion-free of rank 1 but not free.

6. Let $R$ be a commutative ring with 1.
   (a) Show that every proper ideal of $R$ is contained within a maximal ideal.
   (b) Let $J(R)$ denote the intersection of all maximal ideals of $R$. Show that $x \in J(R) \iff 1 + rx$ is a unit for all $r \in R$.
   (c) Suppose now that $R$ is finite. Show that in this case $J(R)$ consists precisely of the nilpotent elements in $R$. (Recall that $x \in R$ is nilpotent if $x^n = 0$ for some positive integer $n$.)

7. Let $p$ be a prime number. Let $A$ be a $p \times p$ matrix over a field $F$, with 1 in all entries except 0 on the main diagonal. Determine the Jordan canonical form (JCF) of $A$
   (a) when $F = \mathbb{Q}$,
   (b) when $F = \mathbb{F}_p$.  

(Hint: In both cases, all eigenvalues lie in the ground field.) In each case find a matrix $P$ such that $P^{-1}AP$ is in JCF.

8. Let $\zeta = e^{2\pi i/8}$.

(a) What is the degree of $\mathbb{Q}(\zeta)/\mathbb{Q}$?
(b) How many quadratic subfields of $\mathbb{Q}(\zeta)$ are there?
(c) What is the degree of $\mathbb{Q}(\zeta, \sqrt[4]{2})$ over $\mathbb{Q}$?