Algebra Qualifying Exam, Spring 2010

1. Let G be a nonabelian finite group. Prove that $|Z(G)| \leq \frac{1}{4}|G|$.

2. Let G be a finite group and let $N \triangleleft G$ be a normal subgroup. Let p be a prime number, and let Q be a subgroup of G such that $N \subset Q$ and Q/N is a Sylow p-subgroup of G/N.

i) Prove that Q contains a Sylow p-subgroup of G.

ii) Prove that every Sylow p-subgroup of G/N is the image of a Sylow p-subgroup of G.

3. Let A be an $n \times n$ matrix over a field F, such that A is diagonalizable. Prove that the following are equivalent:

A) There is a vector $v \in F^n$ such that $v, Av, A^2v, \ldots, A^{n-1}v$ is a basis for F^n.

B) The eigenvalues of A are distinct.

4. Let n be a positive integer and let B denote the $n \times n$ matrix over $\mathbb{C}$ such that every entry is 1. Find the Jordan normal form of B.

5. Determine for which integers n the ring $\mathbb{Z}/n\mathbb{Z}$ is a direct sum of fields. Carefully prove your answer.

6. Let $\zeta \in \mathbb{C}$ be a primitive 12^{th} root of unity.

i) Find the minimal polynomial of ζ over $\mathbb{Q}$.

ii) Show that $\mathbb{Q}[\zeta]$ is Galois over $\mathbb{Q}$, and describe the Galois group of $\mathbb{Q}[\zeta]/\mathbb{Q}$.

iii) Find all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}[\zeta]$. Give your answer in the form $\mathbb{Q}[\alpha]$ for some α, and give proper justification.

7. Let K be a field. Prove that the groups $(K, +)$ and $(K^*, \cdot)$ are not isomorphic. (Hint: Treat $\text{char}(K) = 2$ as a special case.)

8. Let A be a commutative ring with identity. Let $I \subset A$ be an ideal that is maximal among ideals that are not finitely generated. Prove that I is prime.