Preliminary Exam in Complex Analysis

Fall 1992

1. Find a conformal map of the unit disk \( \Delta = \{ z : |z| < 1 \} \) to the first quadrant \( Q = \{ z = x + iy : x > 0, y > 0 \} \) that sends \( z = 0 \) to \( z = 1 + i \).

2. The principal determination of \( z = \text{arctan} w \) is the solution of \( \tan z = w \) for which \( -\pi/2 < \text{Re} z < \pi/2 \). Determine the domain of this principal determination and an expression for the principal determination of \( \text{arctan} w \) in terms of a principal determination of the logarithm function.

3. Let \( w = f(z) \) be a function that is analytic on the closure of the unit disc \( \Delta \) in the complex plane. Assume that \( f \) is not identically zero and show there are points \( z_1, \ldots, z_N \) in \( \Delta \) and numbers \( r_1, \ldots, r_N \) so that

\[
u(z) = \log |f(z)| - \sum_{j=1}^{N} r_j \log |z - z_j|
\]

is harmonic on \( \Delta \setminus \{ z_1, \ldots, z_N \} \).

4. Prove that a non-constant analytic function is an open mapping.

5. Determine (for all values in the domain of \( F \)) the value of

\[
F(w) = \frac{1}{2\pi i} \int_{C} \frac{1}{1- \omega z} dz,
\]

where \( C \) is the positively oriented unit circle \( |z| = 1 \).

6. Evaluate

\[
\frac{1}{2\pi i} \int_{C} \frac{f'(z)}{f(z)} dz
\]

over a (positively oriented) large circle if \( f(z) \) is a polynomial.

7. Give an explicit expression for a meromorphic function \( w = f(z) \) defined on the complex plane whose only poles are simple poles at each point in the set

\[
\mathbb{Z} + i\mathbb{Z} = \{ \omega_{m,n} = m + in : m, n \in \mathbb{Z} \}
\]

and such that the residue of \( f \) is one at each pole. Be sure to state carefully any results used in your construction.

8. Show the function in Problem 7 cannot be doubly periodic. That is show it is impossible that \( f(z + \omega_{m,n}) = f(z) \), for all \( m, n \) in \( \mathbb{Z} \).
9. Let \( g \) be analytic on a disc in the complex plane. Suppose that the differential equation

\[
\frac{dv}{dz} = yg(z)
\]

has an analytic solution in a neighborhood of each point in this disc. Show there is a global analytic solution to this differential equation on this disc.

10. Let \( g = g(z) \) be continuous on \( |z| = 1 \) and define

\[
G(z) = \frac{1}{2\pi i} \int_{|z|=1} \frac{g(\xi)}{\xi-z} d\xi, \quad |z| = 1.
\]

Determine for \( |u| = 1 \)

\[
\lim_{r \to 1} [G(ru) - G(r^{-1}u)].
\]