Complex Analysis Preliminary Exam

Fall 1994

1) Evaluate the following:
   a) \[ \int_{C} \frac{e^z}{z^2 + 1} \, dz \] where \( C \) is the circle of radius 2 centered at 0, oriented counter clockwise.
   b) \[ \int_{0}^{\infty} \frac{x \sin x}{(z^2 + 4)^2} \, dx. \]

2) Give the Laurent expansion for \( f(z) = \frac{(z^2 - 1)}{(z + 2)(z + 3)} \) in the region \( 2 < |z| < 3. \)

3) Let \( f_n(z) = xe^{-\frac{1}{n}z^2}. \) Show this sequence of functions is uniformly convergent on the real axis but is not uniformly convergent on any closed ball, \( B(0, r), \) centered at 0 with radius \( r. \)

4) Let \( \lambda > 1. \) How many roots does \( z + e^{-z} = \lambda \) have in the right half plane?

5) a) Find a conformal map from the infinite strip \( 0 < \text{Im} z < 1 \) onto the semi-infinite strip \( -\frac{\pi}{2} < \text{Re} z < \frac{\pi}{2}, \text{Im} z > 0. \)
   b) Find an harmonic function \( u(z) \) on the semi-infinite strip \( -\frac{\pi}{2} < \text{Re} z < \frac{\pi}{2}, \text{Im} z > 0 \) with boundary values \( u(z) = 1 \) for \( \text{Im} z = 0 \) and \( u(z) = 0 \) for \( \text{Re} z = \frac{\pi}{2} \) and \( \text{Re} z = \frac{\pi}{2}. \)

6) Assume \( 2z(1 - z)\phi'(z) = \phi(z) + z \) and \( \phi(0) = 0. \) Show

   \[ \phi(z) = z + \frac{2}{3}z^2 + \frac{2 \cdot 4}{3 \cdot 5}z^3 + \cdots \]

   for \( |z| < 1. \)

7) Let \( \Omega \) be the unbounded region of the extended plane which is exterior to the two circles of radius 4, centered at 5 and -5. Find a fractional linear transformation mapping \( \Omega \) to an annulus \( 1 < z < R. \) What is \( R? \)
(8) Let \( f(z) \) be analytic in \( \Re z > 0 \) and assume
a) \( f(1) = 1 \),
b) \( f(z + 1) = zf(z) \),
c) \( \frac{d^2}{dz^2} (\log f(z)) = \sum_{n=0}^{\infty} \left( \frac{1}{n + z} \right)^2 \). Prove \( f(z) = z^{-1} e^{cz} \prod_{n=1}^{\infty} e^{z/n} (1 + \frac{z}{n})^{-1} \).

9) Let \( f(z) \) be analytic and assume \( f(0) \neq 0 \) and \( |f(z)| \leq M \) on the circle \( |z| \leq R \). Prove that the number of zeros \( f(z) \) has in the region \( |z| \leq \frac{1}{2} R \) does not exceed \( \frac{1}{\log^2 \log(M/|f(0)|)} \).

10) Classify the one-to-one analytic functions \( f : \mathbb{C} \to \mathbb{C} \). (Sketch the proofs of the theorems you use.)