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Complex Analysis Preliminary Exam

Fall 1994

1) Evaluate the following:

   a) $\int_C \frac{e^z}{z^2+1}dz$ where C is the circle of radius 2 centered at 0, oriented counter clockwise.

   b) $\int_0^\infty \frac{x \sin x}{(z^2+4)^2}dx$.

2) Give the Laurent expansion for $f(z) = \frac{(z^2-1)}{(z+2)(z+3)}$ in the region $2 < |z| < 3$.

3) Let $f_n(z) = ze^{-\frac{1}{2}n^2z^2}$. Show this sequence of functions is uniformly convergent on the real axis but is not uniformly convergent on any closed ball, $B(0,r)$, centered at 0 with radius r.

4.) Let $\lambda > 1$. How many roots does $z + e^{-z} = \lambda$ have in the right half plane?

5) a) Find a conformal map from the infinite strip $0 <$ im$z < 1$ onto the semi-infinite strip $\frac{-\pi}{2} <$ Rez $< \frac{\pi}{2}$, im$z > 0$.

   b) Find an harmonic function $u(z)$ on the semi-infinite strip $\frac{-\pi}{2} <$ Rez $< \frac{\pi}{2}$, im $z > 0$ with boundary values $u(z) = 1$ for im$(z) = 0$ and $u(z) = 0$ for Rez $= \frac{-\pi}{2}$ and Rez $= \frac{\pi}{2}$.

6) Assume $2z(1-z)\phi'(z) = \phi(z) + z$ and $\phi(0) = 0$. Show

$$\phi(z) = z + \frac{2}{3}z^2 + \frac{2\cdot 4}{3\cdot 5}z^3 + \cdots$$

for $|z| < 1$.

7) Let Ω be the unbounded region of the extended plane which is exterior to the two circles of radius 4, centered at 5 and -5. Find a fractional linear transformation mapping Ω to an annulus $1 < z < R$. What is R?

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(8) Let $f(z)$ be analytic in Re$z > 0$ and assume

a) $f(1) = 1$,

b) $f(z+1) = zf(z)$,

c) $\frac{d^2}{dz^2}(\log f(z)) = \sum_{n=0}^{\infty}(\frac{1}{n+z})^2$. Prove $f(z) = z^{-1}e^{cz}\prod_{n=1}^{\infty} e^{z/n}(1+\frac{z}{n})^{-1}$.

9) Let $f(z)$ be analytic and assume $f(0) \neq 0$ and $|f(z)| \leq M$ on the circle $|z| \leq R$. Prove that the number of zeros $f(z)$ has in the region $|z| \leq \frac{1}{3}R$ does not exceed $\frac{1}{\log 2}\log(\frac{M}{|f(0)|})$.

10) Classify the one-to-one analytic functions $f : \mathbb{C} \to \mathbb{C}$. (Sketch the proofs of the theorems you use.)