Complex Analysis Qualifying Exam, January 2011

(6 problems, 2 hours)

1. Include brief justification for your answers to Parts b) and c) of this problem.
   a) Complete the definition: $f : \mathbb{R}^n \to \mathbb{R}^m$ is *(real) differentiable* at a point $a \in \mathbb{R}^n$ if there is a linear transformation $\ldots$.
   b) Give an example of a function $g : \mathbb{R}^2 \to \mathbb{R}$ whose first order partial derivatives exist everywhere, such that g is not differentiable at $(0, 0)$.
   c) Give an example of a function $h : \mathbb{R}^2 \to \mathbb{R}^2$ which is differentiable everywhere, but not complex differentiable anywhere.

2. Find all meromorphic functions that have a simple pole at the point 2 and a double pole at ∞, but are analytic elsewhere. Prove that your list is complete.

3. Write $\mathbb{D}$ for the open unit disc and set $G = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } |z - 1| > 1\}$. Find all conformal one-to-one maps of G onto $\mathbb{D}$. You may express the maps as compositions, but should explain why your list is complete.

4. Use methods of complex variables to evaluate the integral $\displaystyle\int_0^\infty \frac{\cos x - \cos 4x}{x^2}\, dx$.

5. Suppose f is entire and has Taylor series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ about 0.
   a) Express (no proof required) a_n as a contour integral along the circle of radius R centered at 0.
   b) Apply Part a) to verify that the power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on each bounded subset of $\mathbb{C}$.
   c) Determine, with proof, those functions f for which the power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on all of $\mathbb{C}$.

6. Set $H_+ = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$. Suppose $f : H_+ \cup \mathbb{R} \to \mathbb{C}$ satisfies the following:
   (i) $f(i) = i$,
   (ii) f is continuous,
   (iii) f is analytic on H_+,
   (iv) $f(z)$ is real if and only if z is real.

   Show that $f(H_+)$ is a dense subset of H_+.