Answer eight of the following ten questions.

1. Let $G$ be the subgroup $\{ e^{it} \mid t \in \mathbb{R} \}$ of the multiplicative group $\mathbb{C} - \{0\}$.

   (a) Show that the subset $H$ of $G$ formed by all the elements of finite order is an infinite multiplicative group with infinite exponent.

   (b) Show that $H$ is isomorphic to the (additive) group $\mathbb{Q}/\mathbb{Z}$.

   (c) Show that any finite subgroup of $H$ (or equivalently of $\mathbb{Q}/\mathbb{Z}$) is cyclic.

2. Let $p$ be an odd prime number and consider the set $G = \{(x, y, z) \in (\mathbb{Z}/p\mathbb{Z})^3\}$. Using the usual addition and multiplication in $\mathbb{Z}/p\mathbb{Z}$, define the composition law $*$ on $G$:

   $$(x, y, z) * (x', y', z') = (x + x', y + y', xy' + z + z')$$

   for all $(x, y, z), (x', y', z') \in G$.

   (a) Show that $G$ is isomorphic to a Sylow $p$-subgroup of the (multiplicative) group $\text{GL}_3(p)$ of invertible $3 \times 3$ matrices with coefficients in the field $\mathbb{Z}/p\mathbb{Z}$ via the map:

   $$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

   (b) Find the center and the commutator subgroups of $G$.

3. Let $R$ be a commutative ring (with identity 1) and let $I$ be an ideal of $R$ that is contained in all the maximal ideals of $R$. Show that $1 - x$ is a unit, for all $x \in I$.

4. Let $R$ be the ring $\mathbb{Z}[i]$ of Gaussian integers.

   (a) Factorize $9 + 19i$ as product of irreducibles of $R$.

   (b) Determine the field of fractions $F$ of $R$.

   (c) Show that $f = 3X^3 - 6X^2 + 12X - 6$ is irreducible in $F$ but not in $R$.

5. Let $R$ be a (non necessarily commutative) ring (with identity 1) and let $e \in R$ be a central idempotent, that is $e^2 = e$ and $e$ is an element of the center of $R$. Let $M$ be a left $R$-module and set $eM = \{ e \cdot x \mid x \in M \}$ and $(1 - e)M = \{ (1 - e) \cdot x \mid x \in M \}$.

   (a) Show that $eM$ and $(1 - e)M$ are $R$-submodules of $M$.

   (b) Show that there is a split short exact sequence:

   $$0 \longrightarrow eM \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} (1 - e)M \longrightarrow 0$$

   of left $R$-modules, where $f$ is the inclusion and $g$ is the left multiplication by $(1 - e)$.
6. Let $R$ be an integral domain in which there exists no sequence $(a_n)_{n \in \mathbb{N}}$ such that $a_{n+1}$ is a proper factor of $a_n$ (that is $a_{n+1}$ divides $a_n$ and $a_{n+1} \neq a_n$). Prove that $R$ is a unique factorization domain (UFD) if and only if any irreducible element of $R$ is also prime.

7. Let $F$ be a field, $E$ be a field extension of $F$, and $K$ be a field extension of $E$.

(a) Show that if the extension $K/F$ is separable then the extensions $E/F$ and $K/E$ are also separable.

(b) Is the converse true? (no proof required)

8. Let $k$ be a field and let $V$ be a finite dimensional $k$-vector space. Denote by $V^*$ the dual of $V$, that is the set of all homomorphisms of $k$-vector spaces from $V$ to $k$.

(a) Show that the following map, defined on a set of generators of $V^* \otimes V$, extends to a surjective homomorphism of $k$-vector spaces:

$$t : V^* \otimes V \rightarrow k, \quad t(\varphi \otimes v) = \varphi(v)$$

(b) Show that $t$ admits a right inverse if and only if the characteristic of $k$ is either zero, or does not divide the dimension of $V$.

9. Alex, Bart and Carl do their laundry at the same location. Alex washes his clothes once every 11 days, Bart one Friday each 2 weeks, and Carl once every 5 days. Last time that Alex did his laundry was on December 29, 2004; whereas for Bart it was on Friday December 31, 2004; and for Carl it was on January 1st, 2005.

After how many days will/did all 3 of them wash their clothes on the same day, for the first time after January 1st, 2005? (that is 01/01/05 is day 1)

10. Let

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 0 & -1 & -4 \\ -4 & 0 & 0 & 3 \end{pmatrix} \in M_4(\mathbb{R})$$

(a) Find the Jordan form $J$ of $A$.

(b) Find an invertible matrix $P$ such that $J = P^{-1}AP$. (Note that you do not need to compute $P^{-1}$.)

(c) Find the minimal polynomial of $A$. 