Analysis Preliminary Examination — Fall 2000

Show work and carefully justify/prove your assertions.

1. (1) Let \( p^* \) be a limit point of a subset \( A \) of \( \mathbb{R}^N \). Show that each ball \( B(p^*, r) \) (where \( r > 0 \)) around \( p^* \) contains infinitely many points of \( A \);
(2) Let \( A \) and \( B \) be two disjoint compact subsets of a metric space with metric \( d \). Show that there exist \( a \in A, \ b \in B \) such that

\[
d(a, b) = \inf\{d(x, y) \mid x \in A, y \in B\}.
\]

2. Let \( R(x) \) be the function on \([0, 1]\) defined by

\[
R(x) = \begin{cases} 
0 & \text{if } x \text{ is irrational} \\
\frac{1}{n} & \text{if } x = \frac{m}{n} \text{ is rational (integers } m \text{ and } n \text{ have no common factors)}
\end{cases}
\]

Show that \( R(x) \) is Riemann integrable on \([0, 1]\) and find the value of the integral.

3. Let \( f(x) \) be differentiable on \([0, 1]\). Show that its derivative \( f'(x) \) is measurable on \([0, 1]\).

4. Let \( f(x) \in L^1(\mathbb{R}) \) and let \( g(x) \) be a function on \( \mathbb{R} \) with continuous first order derivative. Suppose that \( g(x) \) vanishes outside a bounded closed interval. Define a new function \( h(x) \) by

\[
h(x) = \int_{\mathbb{R}} f(x - t) g(t) dt.
\]

Show that \( h(x) \) is differentiable on \( \mathbb{R} \).

5. Let \( C[0, 1] \) be the space of all complex valued continuous functions on the unit interval endowed with the norm

\[
\|f\|_\infty = \sup_{t \in [0, 1]} |f(t)|.
\]

Let \( C^1[0, 1] \) be the space of all complex valued functions with continuous first order derivative, endowed with the norm

\[
\|f\| = \|f\|_\infty + \|f'\|_\infty.
\]

Let \( B \) be the unit ball in \( C^1[0, 1] \). Show that the closure of \( B \) in \( C[0, 1] \) is compact.

Hint: What are some equivalent conditions for compact sets in a metric space?

6. (1) Let \( z_k \) (\( k = 1, \cdots, n \)) be complex numbers lying on the same side of a straight line passing through the origin. Show that

\[
z_1 + z_2 + \cdots + z_n \neq 0, \quad 1/z_1 + 1/z_2 + \cdots + 1/z_n \neq 0.
\]

(2) Let \( f(z) = z + 1/z \). Describe the images of both the circle \( |z| = r \) of radius \( r \) (\( r \neq 0 \)) and the ray \( \text{arg } z = \theta_0 \) under \( f \) in terms of well known curves.
7. Let $f$ be analytic on a region $R$. Suppose $f'(z_0) \neq 0$ for some $z_0 \in R$. Show that if $C$ is a circle of sufficiently small radius centered at $z_0$, then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$ 

8. Let $A$ be the intersection of the disk $|z + i| < \sqrt{2}$ with the open upper half plane. Find a bijective conformal map from $A$ to the unit disk.

9. Consider a one-to-one analytic mapping $w = u + iv = f(z) = \sum_{n=0}^{\infty} c_n z^n$ from the open unit disk $|z| < 1$ in the $xy$-plane ($z = x + iy$) to a region in the $uv$-plane. For $0 < r < 1$, let $D_r$ be the disk $|z| < r$.

1. Show that the area $\int_{f(D_r)} dudv$ of $f(D_r)$ is finite and is given by $\pi \sum_{n=1}^{\infty} \frac{n|c_n|^2 r^{2n}}$.
2. Give an example of $f$ that is analytic in $|z| < 1$ but the area of $f(D_1)$ is infinite.

10. Let $f(z)$ be an analytic function on $\mathbb{C} \setminus \{z_0\}$, where $z_0$ is a given number. Assume that $f(z)$ is bijective from $\mathbb{C} \setminus \{z_0\}$ onto its range, and that $\lim_{r \to \infty} f(z)$ exists and is finite. Show that $f(z)$ is a fractional linear transformation. Hint: Consider the Laurent series expansion of $f(z)$. 