PH.D. PROBABILITY PRELIMINARY EXAMINATION

Fall 2002

(1) (a) Quote, without proof, the Kolmogorov zero-one law.

(b) Let \( \{a_n\} \) be any sequence of real numbers and \( \{X_n\} \) be a sequence of independent random variables taking the values \( \pm 1 \) with equal probability. Show that the convergence set \( C := \{\omega : \sum a_n X_n \text{ converges}\} \) is a tail event.

(2) (a) Let \( \{X_n, n \geq 1\} \) be a sequence of pairwise independent random variables with \( P\{X_n = n^\delta\} = \frac{1}{2} = P\{X_n = -n^\delta\} \), where \( 0 < \delta < \frac{1}{2} \). Does the law of large numbers (weak or strong) hold for this sequence?

(b) Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables with \( P\{X_n = \sqrt{n}\} = \frac{1}{2} = P\{X_n = -\sqrt{n}\} \). Does the weak law of large numbers hold for this sequence?

(3) Let \( f : (0, \infty) \to \mathbb{R} \) be a bounded continuous function. Prove that the limit relation
\[
\lim_{n \to \infty} \sum_{k=1}^{\infty} f \left( x + \frac{k}{n} \right) \exp\left( -nh \right) \frac{(nh)^k}{k!} = f(x + h),
\]
holds for all \( h > 0, x > 0 \). [Hint: Independent Poisson r.v.s.]

(4) Let \( F_n, F, n \geq 1 \), be distribution functions such that \( F_n \) converges weakly (i.e., in distribution) to \( F \) as \( n \to \infty \).

(a) If the function \( g : \mathbb{R} \to \mathbb{R} \) is uniformly integrable with respect to \( F_n \), show that \( \int g \, dF_n \to \int g \, dF \).

(b) If \( \int |g| \, dF_n \to \int |g| \, dF < \infty \), show that \( g \) is uniformly integrable.

(5) Let \( \{X_n\}, n \geq 1 \), be a sequence of independent random variables and \( \{a_n\}, n \geq 1 \), be a sequence of real numbers such that \( P(X_n = a_n) = \frac{1}{2} = P(X_n = -a_n) \), \( n \geq 1 \). Find conditions on \( \{a_n\} \) so that the sequence \( \{X_n\} \) will satisfy the central limit property.

(6) (a) Let \( \{X_n\}, n \geq 1 \), be a sequence of nonnegative and uniformly bounded random variables adapted to an increasing sequence \( \{\mathcal{F}_n, n \geq 1\} \) of sub \( \sigma \)-algebras. Show that the series
\[
\sum_{n=1}^{\infty} X_n, \quad \text{and} \quad \sum_{n=1}^{\infty} E\{X_n \mid \mathcal{F}_{n-1}\},
\]
where $\mathcal{F}_0$ is the trivial $\sigma$-algebra, either both converge a.s. or both diverge a.s..

(b) Let $\{\mathcal{F}_n, n \geq 1\}$ be an increasing sequence of $\sigma$-algebras and $A_n \in \mathcal{F}_n$, $n \geq 1$ and write $p_1 = P(A_1)$ and $p_n = P\{A_n | \mathcal{F}_{n-1}\}$. Show that

$$P(\limsup_n A_n) = 1 \quad \text{if and only if} \quad P\left\{\omega : \sum_{n=1}^{\infty} p_n(\omega) = \infty\right\} = 1.$$ 

(7) Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d random variables with zero mean and unit variance, $S_n := X_1 + \cdots + X_n$, $n \geq 1$, and let $\{\nu_n, n \geq 1\}$ be a sequence of positive integer valued random variables converging to $\infty$ in probability as $n \to \infty$. Assume that the two sequences $\{X_n, n \geq 1\}$ and $\{\nu_n, n \geq 1\}$ are independent of each other. Show that

$$\lim_{n \to \infty} P\{S_{\nu_n} < x\} = N(0, 1),$$

where $N(0, 1)$ is the standard normal distribution. [Hint: You may need Kronecker lemma.]