Directions: Do all the problems. Problems #1-6 are each worth 10 points; #7 and #8 are each worth 20 points.

1. Give a self-contained proof of the following:
   Let $X$ be a compact metric space. Given any open covering $\mathcal{U}$ of $X$, prove that there is a real number $\epsilon > 0$ so that for each $x \in X$, there is $U \in \mathcal{U}$ such that $B(x, \epsilon) \subseteq U$.

2. a. Prove that if $Y$ is a retract of the Hausdorff space $Z$, then $Y$ is a closed subspace of $Z$.
   b. Let $J$ be an arbitrary set; endow $Z = \prod_{j \in J} \mathbb{R}$ with the product topology. Prove that if $Y$ is a retract of $Z$, then for every normal topological space $X$, closed subspace $A \subseteq X$ and continuous function $f : A \rightarrow Y$, there exists a continuous extension $\tilde{f} : X \rightarrow Y$.

3. Let $X$ be the set of real numbers endowed with the topology generated by basis elements $[a, b), a, b \in \mathbb{R}, a < b$. Let $\mathbb{R}$ denote the set of real numbers endowed with the standard topology.
   a. Classify all continuous functions $f : \mathbb{R} \rightarrow X$.
   b. Classify all continuous functions $f : X \rightarrow \mathbb{R}$.

4. Prove or give a counterexample in each case: If $X$ is a contractible space, then
   a. $X$ is simply connected.
   b. $X$ is locally simply connected at each point $x \in X$.

5. Let $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and let $S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n : |x| = 1\}$. Suppose $f : D^n \rightarrow \mathbb{R}^n$ is continuous and satisfies $|f(x) - x| < 1$ for all $x \in S^{n-1}$. Prove that $0 \in f(D^n)$.

6. View the torus $T$ as the quotient space $\mathbb{R}^2 / \mathbb{Z}^2$, so we have the obvious covering map $\pi : \mathbb{R}^2 \rightarrow T$. The matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ defines a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$.
   a. Prove briefly that this linear map induces a continuous map $f : T \rightarrow T$.
   b. Prove or disprove: $f$ is homotopic to the identity map.

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7. Let $S^2, S^{2'}$ be two copies of the two-sphere, and let $p, q \in S^2$, $p', q' \in S^{2'}$ be pairs of points in the respective copies. Define

$$X = S^2 \cup S^{2'} / (p \sim p', q \sim q').$$

a. Give the universal covering space of $X$.
b. Using Van Kampen's Theorem, compute $\pi_1(X)$ and relate your answer to your answer to a.
c. Compute $H_*(X, \mathbb{Z})$ by any method you desire. Give details.

8. a. Define the Lefschetz number $L(f)$ of a continuous map $f : X \to X$.
b. Let $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ be continuous. Give (with proof) necessary and sufficient conditions for $f$ to have a fixed point.
c. Let $K$ be a simplicial complex. State and sketch a proof of the Lefschetz fixed point theorem for a continuous map $f : |K| \to |K|$.