• [1] Suppose that a square matrix $A$ is symmetric and positive definite. Show that when applying Gauss-Seidel iteration to solve $Ax = b$, the iteration converges.

• [2] Let $A$ be a tridiagonal matrix. Write $A = QR$ with $R$ being upper triangular matrix and $Q$ orthonormal matrix. Show that $RQ$ is also a tridiagonal matrix.

• [3] Let $f(x)$ be a continuous function over $x \in [0,1]^n$ with $n \geq 2$. Explain how to use the bisection method to find a zero inside $(0,1)$ if $f$ changes sign over some vertices of the cube, say $f(0,0) > 0$ and $f(1,1) < 0$ for $f$ defined over $[0,1] \times [0,1]$.

• [4] If using the following formula to compute an approximation of $f'(x)$:

$$f'(x) \approx \frac{1}{12h}[-f(x + 2h) + 8f(x + h) - 8f(x - h) + f(x - 2h)],$$

find the order of convergence as $h \to 0$.

• [5] Let $\triangle := \{\cdots \leq x_{-1} \leq x_0 \leq x_1 \leq \cdots \}$ be a nondecreasing knot sequence. Suppose that $x_{i+m} - x_i > 0$ for all $i$. Define the $m$th order B-spline functions over $\triangle$ by

$$B^m_i(x) = (x_{i+m} - x_i)[x_i,\ldots,x_{i+m}](x-x_i)^{n-1}.$$

Show that B-splines satisfy $B^m_i(x) \geq 0$ for all $i$ and

$$\sum_{i=k-m+1}^{k-1+j} B^m_i(x) = 1, \quad \forall x \in (x_k,x_{k+j})$$

by using induction.

• [6] Derive the modified Euler’s method:

$$x(t + h) = x(t) + hf(t + h/2, x(t) + f(t, x(t))/2)$$

by performing Richardson’s extrapolation on Euler’s method using step size $h$ and $h/2$. Show that the truncated error is $O(h^2)$. 