Instruction: The following are 8 problems in total. Please start each problem on a separate sheet of paper, write on only one side of the paper, and number each page. The time limit on this exam is three hours.

1. Use Taylor’s theorem for a function of two variables to carefully derive and state Newton’s method for the numerical solution of the system of two nonlinear equations:

\[
\begin{align*}
    f(x, y) &= 0, \\
g(x, y) &= 0.
\end{align*}
\]

Give a sufficient condition which ensures that the Newton method converges.

2. Let \( x \) and \( \tilde{x} \) be the solution of two linear systems: \( Ax = b \) and \( \tilde{A}\tilde{x} = \tilde{b} \). Show
   (a) If \( A = \tilde{A} \), then
   \[
   \frac{\|x - \tilde{x}\|}{\|x\|} \leq \text{cond}(A) \frac{\|b - \tilde{b}\|}{\|b\|}.
   \]
   where \( \text{cond}(A) = \|A\|\|A^{-1}\| \) stands for the condition number of \( A \).
   (b) Consider the case that \( A \neq \tilde{A} \). Show that if \( \|A^{-1}\| \cdot \|A - \tilde{A}\| < 1 \), then
   \[
   \frac{\|x - \tilde{x}\|}{\|x\|} \leq \frac{\text{cond}(A) \left( \frac{\|A - \tilde{A}\|}{\|A\|} + \frac{\|b - \tilde{b}\|}{\|b\|} \right)}{1 - \text{cond}(A) \|A^{-1}\|}.
   \]

3. Recall that for any nonzero vector \( v \) of size \( n \times 1 \), \( I - 2vv^T/\|v\|^2 \) is called a Householder matrix, where \( I \) is the \( n \times n \) identity matrix. Show that there exists a sequence of Householder matrices \( H_1, \ldots, H_n \) which converts any matrix \( A \) into a lower triangular matrix \( L \), that is, \( H_n \cdots H_1 A = L \). (Hint: you may use a matrix of \( 4 \times 4 \) to explain how to do.)

4. Let \( f \) be a continuous function on \( [a, b] \). The following statements are true or false. If it is true, give some reasons, e.g., quote a well-known theorem. If it is false, give an example.
   (1) There exists a sequence of polynomials \( p_n \) such that \( p_n \) converges to \( f \) uniformly.
   (2) There exists a sequence of interpolatory polynomials \( p_n(f) \) which interpolates \( f \) at \( n+1 \) distinct points \( x_i^{(n)} \in [a, b], i = 0, 1, \ldots, n \) such that \( p_n(f) \) converges to \( f \) uniformly.
   (3) Let \( x_i^n = a + i(b - a)/n, i = 0, \ldots, n \). The interpolatory polynomial \( p_n(f) \) at these \( x_i^n \)'s converges to \( f \) pointwisely.
5. Let \( p_n(x) = \sum_{i=0}^{n} c_i B_i^n(x) \) be a polynomial in B-form with respect to \([a, b]\). Here,

\[ B_i^n(x) = \binom{n}{i} \left( \frac{x-a}{b-a} \right)^i \left( \frac{b-x}{b-a} \right)^{n-i} \]

is defined on the interval \([a, b]\). Similarly, let \( q_n(x) = \sum_{i=0}^{n} d_i \tilde{B}_i^n(x) \) with \( \tilde{B}_i^n(x) = \binom{n}{i} \left( \frac{x-b}{c-b} \right)^i \left( \frac{c-x}{c-b} \right)^{n-i} \) defined on \([b, c]\). Derive the conditions on their coefficients of \( p_n \) and \( q_n \) that ensure

\( \frac{d^r}{dx^r} p_n(b) = \frac{d^r}{dx^r} q_n(b) \), \( \forall r = 0, 1, 2. \)

6. Find an approximation for \( \Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \) using the following four function evaluations: \( f(x, y) \), \( f(x, y + 2h) \), \( f(x - \sqrt{3}h, y - h) \) and \( f(x + \sqrt{3}h, y - h) \), which form the three corners and the center of an equilateral triangle as shown below. Note that the points A, B, C and D correspond to the locations of these given function values, respectively.

![Equilateral Triangle with Function Values](image)

7. Show that when the composite trapezoid rule is applied to \( \int_a^b e^x \, dx \) using equally spaced points, the relative error is exactly

\[ 1 - \frac{h}{2} - \frac{h}{e^h - 1} \]

8. Derive the general 2\textsuperscript{nd}-order Runge-Kutta method for numerical solution of ODE, where \( \alpha \) is a variable:

\[
\begin{align*}
K_1 & = h f(t, x(t)) \\
K_2 & = h f(t + \alpha h, x(t) + \alpha K_1) \\
x(t + h) & = x(t) + AK_1 + BK_2
\end{align*}
\]

In other words, express \( A \) and \( B \) in terms of \( \alpha \).