Mathematics Preliminary Exam, Fall 2012

Attempt all problems; they are weighted equally. Write \( \mathbb{N} \) for the set of positive integers and \( \mathbb{R} \) for the set of real numbers.

1. Write the negations of these sentences in as "smooth" a way as possible. (In particular, you may not simply append "It is not the case that ...". Also, you should make explicit any "hidden" quantifiers.)
   (a) There is a real number \( x \) such that for every real number \( y \), \( |x - y| > 1 \).
   (b) A real-valued function that is continuous on a closed interval attains a minimum value on that interval.
   (c) \( 3n + 1 \) is even if and only if \( n^2 + 4 \) is prime.

2. Let \( A = \begin{pmatrix} 7 & -3 \\ 1 & 3 \end{pmatrix} \). Find an invertible matrix \( P \) and a diagonal matrix \( D \) with \( P^{-1}AP = D \).
   [You should not have to compute \( P^{-1} \).]

3. Let \( A \) be an \( m \times n \) real matrix. Write \( A^t \) for the transpose of \( A \), and \( N(A) \) for the nullspace of \( A \). Prove that \( N(A^tA) = N(A) \).

4. Suppose \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( f(xy) = xf(y) + yf(x) \) for all \( x, y \in \mathbb{R} \). Prove that \( f(1) = 0 \) and that \( f(u^n) = nu^{n-1}f(u) \) for all \( n \in \mathbb{N} \) and \( u \in \mathbb{R} \).

5. Give an \( \epsilon-\delta \) proof that \( \lim_{x \to 2} \frac{1}{x^2 + 1} = \frac{1}{5} \).

6. Let \( f : X \to Y \) be a (not necessarily invertible!) function, and \( A \subseteq X \).
   (a) Prove that \( A \subseteq f^{-1}(f(A)) \).
   (b) Prove that if \( f \) is injective (one-to-one) then \( A = f^{-1}(f(A)) \).
   (c) Give an example for which \( A \neq f^{-1}(f(A)) \).

7. Prove that the line integral \( \int_C (x + y^2) \, dx + (e^y + 3xy^2) \, dy \) is path-independent; i.e., it depends only on the endpoints of \( C \).

8. Let \( f_n(x) = \frac{nx}{n + x} \) for \( x \in [0, \infty) \) and \( n \in \mathbb{N} \).
   (a) Find a function \( f \) such that \( \{f_n\} \) converges to \( f \) pointwise on \( [0, \infty) \).
   (b) Is the convergence uniform on \( [0, \infty) \)? Justify your answer.

9. Suppose \( R \) is a commutative ring (with 1), \( I \) is a proper ideal in \( R \), and \( a \in R \). Suppose \( \langle a \rangle + I = R \). Prove that \( a + I \) is a unit (i.e., invertible) in the quotient ring \( R/I \).
   (For half credit: Prove that if \( a \) and \( n \) are relatively prime integers, then \( a + n\mathbb{Z} \) is a unit in \( \mathbb{Z}/n\mathbb{Z} \).)