Preliminary Exam, Spring 2008

(3 hours, 8 problems counted equally)

1. Suppose $f, g, h : A \to A$ are functions. Prove that if f is injective and $f \circ g = f \circ h$, then $g = h$.

2. Prove from the ϵ–δ definition that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is continuous at 2.

3. State both parts of the Fundamental Theorem of Calculus and outline a proof of one of them.

4. Let T be a linear transformation on a finite-dimensional vector space V. Suppose $x_1, x_2, \ldots, x_n$ is a basis for V with the first k of those vectors being a basis for the kernel of T. Prove that $Tx_{k+1}, Tx_{k+2}, \ldots, Tx_n$ is a basis for the range of T.

5. Let $A := \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Find an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal.

6. Let $f : \mathbb{R} \to \mathbb{R}$ have derivatives of all orders and suppose $f(x) = 0$ for each integer x. Prove that for each $n \in \mathbb{N}$, the $n'th$ derivative of f has at least one root.

7. Determine, with proof, which if any of the following functions mapping $\mathbb{R}^2$ into $\mathbb{R}$ are continuous.

a) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & x = y = 0 \end{cases}$

b) $g(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & x = y = 0 \end{cases}$

8. Give examples of the following. No proofs are required.

a) open sentences $P(x)$ and $Q(x)$ such that the statement $\forall x[P(x) \vee Q(x)]$ is true, but the statement $[\forall x P(x)] \vee [\forall x Q(x)]$ is false,

b) a (real) power series whose domain of convergence is the half-open interval $[0, 4)$.

c) a sequence (f_n) of real valued functions from $\mathbb{R}$ to $\mathbb{R}$ which converges pointwise, but not uniformly.