Team Round, 30 minutes

October 26, 2002

WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 50 points, for a total of 200 points.

Problem 1. Find the radius of a sphere inscribed in a regular tetrahedron of side 1.

Answer.

$$\frac{\sqrt{2}}{4\sqrt{3}} = \frac{\sqrt{6}}{12} = \frac{1}{2\sqrt{6}} = \frac{1}{\sqrt{24}}$$

Solution. The radius is 1/4 of the height: this can be seen for example by using weights, since the center of the sphere will be the barycenter of the tetrahedron.

If h is the height then $h^2 = 1^2 - (2/3)s^2$, where s is the height in a regular triangle. Hence, $s^2 = 1^2 - (1/2)^2$. Putting this all together, we obtain

$$h = \frac{1}{4}\sqrt{1^2 - \left(1^2 - \left(\frac{1}{2}\right)^2\right) \times \left(\frac{2}{3}\right)^2}$$

Problem 2. Find

$$\sum_{i=1}^{99} \lfloor 0.37i \rfloor$$

($\lfloor x \rfloor$ denotes the integral part of x.)

Answer. 1782

Solution. We will do this for arbitrary relatively prime numbers a and b (in our example $a = 100$ and $b = 37$). On a grid, draw the $a \times b$ rectangle. Then the sum represents the number of lattice points in the interior $(a-2) \times (b-2)$-rectangle which lie under the diagonal. Since there is the same number of points that lie over the diagonal, the answer is $\frac{(a-1)(b-1)}{2}$. In our case, we get $\frac{99 \cdot 36}{2} = 1782$.

Problem 3. Find the number of even binomial coefficients in the 30th row of Pascal's triangle (the one starting with 1, 30, ...).

Answer. 15

Solution. Let us first find the number of odd binomial coefficients. We need to compute the number of odd coefficients in $(1+x)^{30}$ which is the same as the number of powers of x in $(1+x)^{30} \mod 2$. Note that $(a+b)^2 = a^2 + b^2 \mod 2$, and therefore $(a+b)^4 = a^4 + b^4 \mod 2$ and similarly for any power of 2. We have $30 = 16 + 8 + 4 + 2$. Hence,

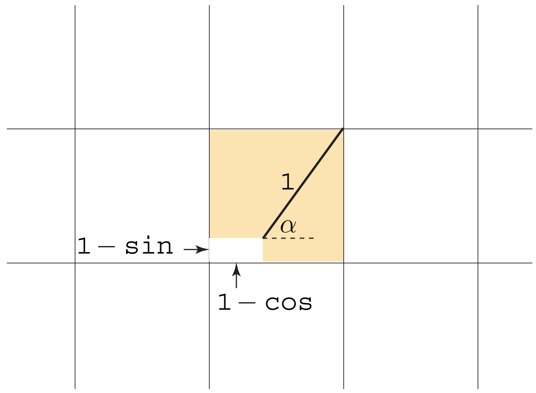
$$\begin{aligned} (1+x)^{30} &= (1+x)^{16}(1+x)^8(1+x)^4(1+x)^2 \\ &= (1+x^{16})(1+x^8)(1+x^4)(1+x^2) \mod 2 \end{aligned}$$

In the last product all monomials are distinct, since every positive integer can be written as a sum of powers of 2 in a unique way. Therefore, there are $2^4 = 16$ odd coefficients and $31 - 16 = 15$ even ones.

Problem 4. Imagine a rectangular grid of lines, horizontal and vertical, 1 meter apart. What is the probability that a pin 1 meter long will touch one of the lines on the grid when dropped at random?

Answer. $\frac{3}{\pi}$

Solution. Suppose the pin is dropped at an angle $0 \leq \alpha \leq \pi/2$. Then it will intersect the grid if and only if the left end falls in the shaded area on the picture below,



whose area is

$$1-(1-\sin\alpha)(1-\cos\alpha)=\sin\alpha+\cos\alpha-(\sin 2\alpha)/2\ .$$

Since different values of α are equally likely, the probability is

$$p=\frac{2}{\pi}\int_0^{\pi/2}(\sin\alpha+\cos\alpha-(\sin 2\alpha)/2)\,d\alpha=\frac{2}{\pi}\cdot\frac{3}{2}=\frac{3}{\pi}$$