University of Georgia High School Math Tournament

1785

2003

Sponsored by: UGA Math Department, UGA Math Club,
UGA Parents and Families Association

Written test, 25 problems / 90 minutes

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately.

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

   where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

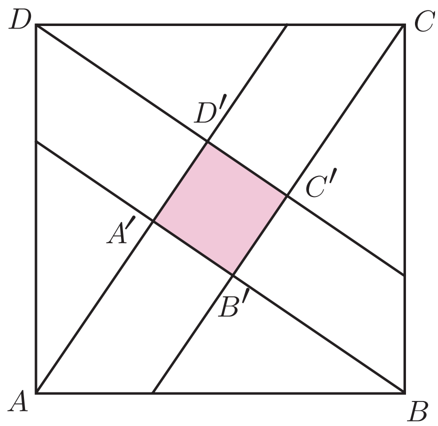
4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit through the rear doors.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

Problem 1. On the picture below (not to scale, adapted from an actual Chinese drawing from 1000 B.C.) the area of the large square $ABCD$ is 25, and the area of the small square $A'B'C'D'$ is 1. Find the length of AA'.



(A) 1 (B) 2 (C) 3 (D) 4 (E) None of the above

Problem 2. Of the first 3,000,000,000 positive integers, what portion is divisible by 2 but not by 3?

(A) 1/6 (B) 1/3 (C) 1/2 (D) 2/3 (E) None of the above

Problem 3. How many two-digit numbers double when the two digits are interchanged?

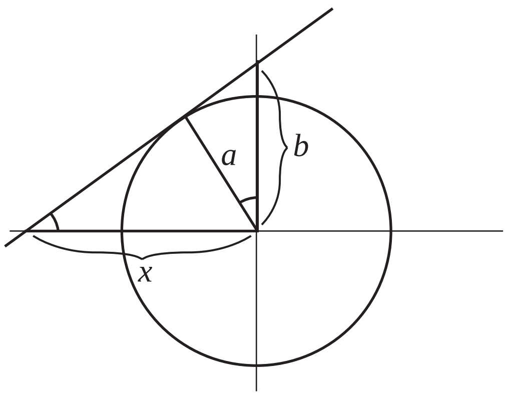
(A) 1 (B) 2 (C) 3 (D) 4 (E) none

Problem 4. Let x be the smallest positive integer which gives remainder 1 when divided by 2, remainder 2 when divided by 3, remainder 3 when divided by 4 and remainder 4 when divided by 5. What is the sum of the digits of x?

(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

Problem 5. If the line $y = mx + b$ with $b > 0$ is tangent to the circle $x^2 + y^2 = a^2$, then

(A) $b = a^2(m^2 + 1)$ (B) $a = b\sqrt{m^2 + 1}$ (C) $ab = m^2 + 1$
(D) $b^2 = a^2(m^2 + 1)$ (E) not enough information



Problem 6. Suppose you are writing positive integers in a row, without blank spaces, like this:

$$123456789101112\ldots$$

What will be the 1000th digit?

(A) 1 or 2 (B) 3 or 4 (C) 5 or 6 (D) 7 or 8 (E) 9 or 0

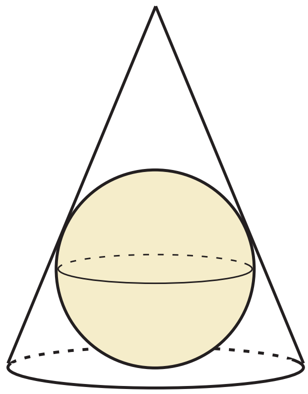
Problem 7. Find

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + 30^2$$

(A) 465 (B) 7855 (C) 9402 (D) 9455 (E) 13505

Problem 8. A sphere is inscribed in a right circular cone with vertex angle $60°$. The ratio of the volume of the sphere to the volume of the cone is

(A) $1/3$ (B) $\sqrt{3}/4$ (C) $4/9$ (D) $\sqrt{2}/3$ (E) $1/2$



Problem 9. Alice, Bob, Charlie, Diane and Ed sit at a round table in random order. What is the probability that Alice and Bob are neighbors?

(A) $1/8$ (B) $1/4$ (C) $1/6$ (D) $1/2$ (E) $2/3$

Problem 10. If $aaa_9 = bbb_{16}$ (the first numeral is in base 9 and the second one is in base 16) then $a/b =$

(A) 1 (B) 2 (C) 3 (D) 4 (E) None of the above

Problem 11. Find $\sqrt{x^2 - y^2}$ if x and y satisfy the following system of equations:

$$\begin{aligned} x + y + \sqrt{x+y} &= 72 \\ x - y - \sqrt{x-y} &= 30 \end{aligned}$$

(A) 24 (B) 30 (C) 48 (D) 72 (E) None of the above

Problem 12. Among the following shapes of equal area, which one has the largest perimeter?

(A) circle (B) triangle (C) square (D) regular pentagon
(E) regular hexagon

Problem 13. Only one of the following numbers is prime. Which one?

(A) 19972003 (B) 19992003 (C) 20012003 (D) 20022003
(E) 20032003

Problem 14. Let us play the following game. You have \$1. With every move, you can either double your money or add \$1 to it. What is the smallest number of moves you have to make to get to \$200?

(A) 6 (B) 7 (C) 8 (D) 9 (E) It is impossible to get to \$200

Problem 15. You repeatedly throw a coin. What is the probability that heads comes up three times before tails comes up twice?

(A) 1/16 (B) 3/16 (C) 5/16 (D) 1/2 (E) None of the above

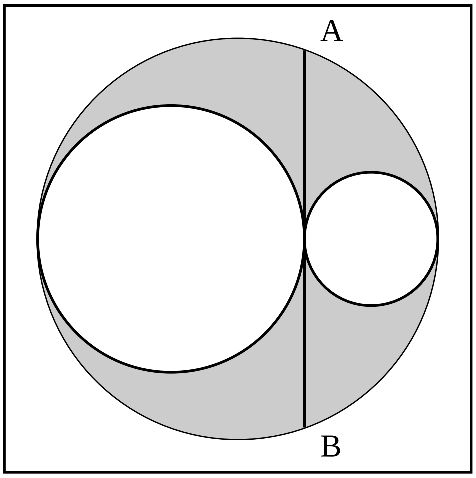
Problem 16. A ball is shot from a corner of a square billiard table with a side 1. It bounces 3 times off the walls and then falls into a corner. What is the greatest distance it could have possibly traveled?

(A) $\sqrt{13}$ (B) 4 (C) $\sqrt{17}$ (D) 5 (E) None of the above

Problem 17. If $a + b = 1$ and $a^3 + b^3 = 4$, then $a^4 + b^4 =$

(A) 1 (B) 3 (C) 7 (D) 9 (E) none of the above

Problem 18. The length of the chord AB is 4. Find the area of the shaded region.



(A) $\pi/2$ (B) π (C) $\pi + 1$ (D) 2π (E) None of the above

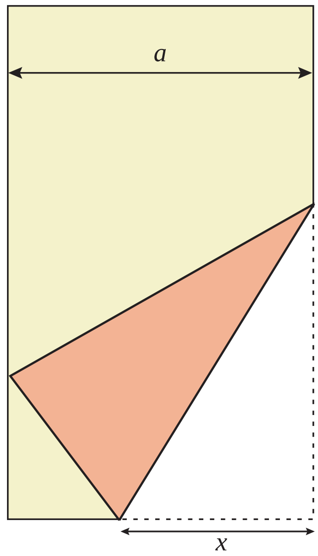
Problem 19. There are 120 permutations of the word **BORIS**. Suppose these are arranged in alphabetical order, from BIORS to SROIB. What will be the 60th permutation?

(A) ORSIB (B) OSBIR (C) OISRB (D) OBSIR (E) OIBRS

Problem 20. How many times during a 24-hour day are the hour hand and the minute hand of a watch perpendicular to each other? (For example, this is true at 3 a.m.).

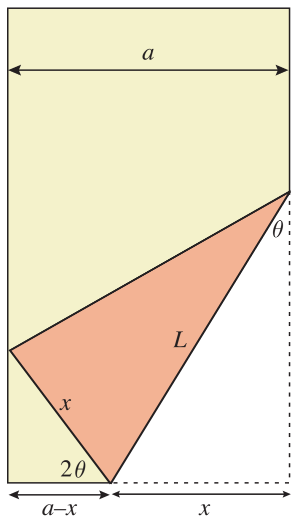
(A) 4 (B) 22 (C) 24 (D) 44 (E) 48

Problem 21. A (very long) piece of paper is folded, as pictured, bringing the right bottom corner to the left edge of the paper. If the width of the paper is a, and the length folded over is x, as marked in the picture, then the length of the crease is



(A) $\dfrac{ax\sqrt{2}}{2x-a}$ (B) $x\sqrt{2}$ (C) $\dfrac{x\sqrt{2a}}{\sqrt{2a-x}}$

(D) $x^2\sqrt{6}\sqrt{4x^2-a^2}$ (E) $\sqrt{\dfrac{2x^3}{2x-a}}$



Problem 22. Among the numbers 2000, 2001, 2002, 2003, how many can be written in the form $n^2 + m^2$ for some integers n and m?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 23. What is the number of pairs (x, y) of integers satisfying

$$x^2 + y^2 \leq 100\,?$$

(A) 101 (B) 179 (C) 297 (D) 317 (E) 361

Problem 24. Find out how many numbers in the 100th row of the Pascal triangle (the one starting with 1, 100, ...) are not divisible by 3.

(A) 4 (B) 12 (C) 27 (D) 32 (E) None of the above

Problem 25. Among the first one billion positive integers, consider the sets of:

(1) palindromic numbers (such as 22, 121, 11533511, etc.),

(2) prime numbers (such as 2, 3, 5, 7, etc.), or

(3) perfect cubes (such as 1, 8, 27, 64, etc.).

Arrange these in the order of decreasing size.

(A) 1,2,3 (i.e. palindromic numbers are the most frequent, then primes, then cubes) (B) 1,3,2 (C) 2,1,3 (D) 2,3,1 (E) 3,1,2

Authors. Written by Valery Alexeev, Boris Alexeev and Ted Shifrin ©2003. Problem 8 is taken from the June 1997 issue of $\mathbb{Z}$imaths published by the University of Zimbabwe.