Sponsored by: UGA Math Department and UGA Math Club

TEAM ROUND / 1 HOUR / 210 POINTS
October 2, 2010

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

Problem 1 (Prüfer Airlines). In a certain small European country there are only 5 cities. An airline decides to connect them by 4 (two-way) connections, so that it would be possible to fly from any city to any other, possibly with stops.

(a) (35 points) In how many ways is it possible to do this?

(b) (35 points) Same question but for 6 cities and 5 connections.

Problem 2 (Let's be friends). One hundred (100) people go through the following procedure. One-by-one, they each randomly point at a person who is not yet pointed at. A person may point at himself, so for example, the first person points at himself with probability 1%.

What is the probability that after this procedure there exist 75 people $P_1, P_2, \ldots, P_{75}$ so that person P_1 points at P_2, person P_2 points at P_3, ..., person P_{74} points at P_{75}, and finally person P_{75} points at P_1?

Problem 3 (A very fair division). It is possible to divide the integers $1, 2, \ldots, 8$ into two sets A and B in a unique manner so that

- 1 is in A,

- A and B contain the same number of elements, and
- the sum of the elements in A equals the sum of the elements in B, and
- the sum of the squares of the elements in A equals the sum of the squares of the elements in B.

It is also possible to divide the integers $1, 2, \ldots, 16$ into two sets A and B in a unique manner so that all of the above hold, *as well as*

- the sum of the *cubes* of the elements in A equals the sum of the *cubes* of the elements in B.

This is a two-part problem:

(a) (35 pts) What is the set A in the case of 8 numbers?

(b) (35 pts) What is the set A in the case of 16 numbers?

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Team ID:

Team name:

Answer 1:
(a)
(b)

Answer 2:

Answer 3:
(a)
(b)